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PREFACE

The aim of this text is to give just those portions of Algebra which are usually taught in the ordinary course in College Algebra in High Schools, Colleges, and Technical Schools, and to omit all superfluous matter.

The first chapter contains review problems, without subject-matter, covering the whole ground of Elementary Algebra, for use in classes which are found to need such a review. The text proper begins with the subject of Radicals. It has been the experience of the authors that the average student, who has had only a year and a half of Algebra, enters college with a very limited knowledge of Radicals, Theory of Exponents, and Quadratics; and as these subjects are most essential to his future success, a thorough review is needed. They have, therefore, been treated very fully.

Some subjects which formerly found a place in College Algebra, such as Series, Continued Fractions, Differentiation, etc., have been omitted, either because they are too difficult or of little value. Graphical representation of equations is treated fully, being applied both to quadratics and to equations of higher degree. This will be found to interest the student and at the same time to be of great value in his future work. Attention is called to the simplicity of the treatment of Determinants, which is somewhat different from the method usually followed. The Theory of Equations is treated quite fully, yet only so much of the subject is given as will enable the student to solve readily any numerical equation.

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*** Teachers may obtain answer-books, for which no charge is made, on application to the publishers.

COLLEGE ALGEBRA

CHAPTER I

REVIEW EXERCISES

EXERCISE I

Simplify the following expressions:

1. $(a-b)-[(c-d)-(e-f)]+(g-h)$.
2. $(a-b)+c-[(d+e)-f-(g-h)]$.
3. $a-[b-(c-d)]-[e+(f-g)-h]$.
4. $a-[b-(c+d)+e]-(f-g)+h$.
5. $[(a-b)+(c-d)]-[(e+f)+(g-h)]$.
6. $[(a+b)-(c+d)]+[(e-f)-(g+h)]$.
7. $[(a-b)-(c-d)]-[(e-f)-(g-h)]$.
8. $[(a+b)+(c-d)]+[(e-f)-(g-h)]$.
9. $(3x+5y)-[(7x-2y)-(8x-4y)]+(x-y)$.
10. $(7m-4)+3p-[(8g+3p-2)+5m-(3g-p)]$.
11. $a-[b-(c-[d-(e-f)-g]+h)-k]$.
12. $a-[2a-(\{3a-7a\}-3c)]$.
13. $a-[-(-\{-3a-(2a-b)\})]$.
14. $m-[-n-\{-3n-(4m-6n)\}]$.
15. $a-[\{b-(c+d)\}+\{e+(f-g+h)-(k+l-m)\}$ $-(n-v)]$.

16. $a-(b+\{c+2x\}-\{y-z\})$.

17. $x-(2x-y-[3x-2y-(4x-3y)])$.

18. $a[a+(b-c)]\div(a+b-c)$.

19. $5x-(3x-4)-[7x(2-9x)]$.

20. $(a^2+ab+b^2)(a-b)-3(a^3-b^3)$.

21. $7a-2[3a-2b+\{(a+b)-(a-b)\}]$.

22. $[-c+(a+b)][-c-(a+b)]-(c^2-a^2)\div(c+a)$.

23. $(a-b)(a+b-c)+(b-c)(b+c-a)$
$+(c-a)(c+a-b)$.

24. $(a+b)(a-b)-\{(a+b-c)-(b-a-c)$
$+(b+c-a)\}\{a-b-c\}$.

25. $(x^2-y^2)\div(x+y)+3[(x-y)(x^2-2xy+y^2)]$
$\div[(x-y)(x-y)(x-y)]$.

26. $8a\div4a+7-[6a\div2]\cdot3-6a^2\div2a-7(1-a)$.

27. $(x^2-y^2-z^2+2yz)\div(x-y+z)$.

28. $3a-[b-a-4(2a+b-\{a-b\})]$.

29. $a-2[2a-b-(3a-2b-\{4a-3b\})]$.

30. $(x^2-2x+1)(x^3-3x^2+3x-1)$.

31. $(x^5-32)\div(x^4+2x^3+4x^2+8x+16)$.

32. $(15x^4+7x^3+15x^2+7x+4)\div(3x^2+2x+1)$.

33. $(x^5-y^5)\div(x^4+x^3y+x^2y^2+xy^3+y^4)$.

34. $(1+2p+4p^2+8p^3+16p^4)(1-2p)$.

35. $(216a^3+125)\div(36a^2-30a+25)$.

36. $(x^2+y^2-z^2)(x^2+y^2+z^2)-(x^2-y^2+z^2)(x^2-y^2-z^2)$.

37. $(2x^2-3xy+y^2-3)(3-y^2-3xy+2x^2)$.

38. $(77a^9b^3-55a^8b^2-35a^8b+25a^7)\div(5a-7a^2b)$.

EXERCISE II

Write the indicated products by inspection:

1. $(2a + b)^2$.
2. $(x - 11)(x - 6)$.
3. $(1 - 2x - 3y)^2$.
4. $(a + 7)(a - 7)$.
5. $(x - 3)(x^2 + 3x + 9)$.
6. $(1 - 4x)^3$.
7. $(2x + y + 1)(2x + y - 1)$.
8. $(3x^2 - 5y^2)^2$.
9. $(3x^2 + 4xy^2)(3x^2 - 4xy^2)$.
10. $(2m - 3n)(3m - 4n)$.
11. $(x + 4)(x^2 - 4x + 16)$.
12. $(2m^2 - n^2 - p^2)(2m^2 - n^2 + p^2)$.
13. $(2m^2 - n^2 - p^2)(2m^2 - n^2 - p^2)$.
14. $(2x^2 - 5y^2)(5x^2 - 2y^2)$.
15. $(14 - a)(3 - a)$.
16. $(1 - 4xyz)^3$.
17. $(2x^2y^2 - z^2)^2$.
18. $(3x - a)(9x^2 + 3xa + a^2)$.
19. $(7a^2 + 3xy)(7a^2 + 3xy)$.
20. $(5xy - 3ab)(5xy + 3ab)$.
21. $(5xy + 1)(25x^2y^2 - 5xy + 1)$.
22. $(3m - 2n + y)(3m + 2n - y)$.
23. $(a^2 + ax + x^2)(a^2 - ax + x^2)$.
24. $(a + x + y)(a - x - y)$.
25. $(ax + m - n)(ax - d)$.
26. $(m - n - x + y)(m - n + x - y)$.
27. $(2ax - 3by)(4a^2x^2 + 6abxy + 9b^2y^2)$.

✓**28.** $(1+x+x^2)(1-x+x^2)$.

29. $(1+x+x^2)(1+x+x^2)$.

30. $(1+x+x^2)(1+x-x^2)$.

✓**31.** $[(m+2n)+(a+b)]^2$.

32. $(7m^2a^2-4d^2)^2$.

33. $(2a^2+7b)(2a^2-7b)$.

✓**34.** $[(2a+3b)+(3c+d)]^2$.

35. $\{2(a-b)-c\}^2$.

36. $(1-5x+2x^2)^2$.

✓**37.** $(2a^2-3ab+4)(-2a^2-3ab+4)$.

38. $(9x^2y^2-7xyz^2)^2$.

39. $(7x^2y^2z-4xyz^3)^3$.

✓**40.** $\{2(a-b)-3(c+d)\}^2$.

41. $(x^2-y^2+3)(x^2-y^2-11)$.

42. $\{3(a+b)+5m^2(a^2+b^2)\}^2$.

✓**43.** $(2m-4n)(3m+n)$.

44. $(a+b-c^2)(a+b+c^2)$.

45. $(5ny-cz)(7ny-4cz)$.

✓**46.** $(a^2-3b^2)(a^4+3a^2b^2+9b^4)$.

47. $\{(2a+b)+3c^2\}\{(2a+b)-3c^2\}$.

48. $[x-y+3a][x-y-2a]$.

✓**49.** $[2(a-b)+3c][2(a-b)-2c]$.

50. $(2c+2d+5a+5b)(2c+2d-5a-5b)$.

51. $[3(a+b)-2c][2(a+b)+5c]$.

✓**52.** $(x^2-xy-z^2)^3$.

53. $[a-3(x-y)][2a-4(x-y)]$.

54. $\{13(a+b)-5(a^2+b^2)\}\{13(a+b)+5(a^2+b^2)\}$.

✓**55.** $[7(m-n)-(a-b)][3(m-n)+4(a-b)]$.

EXERCISE III

Factor:

1. $x^2 - 144$.
2. $16a^2 - 121b^2$.
3. $1 - 100c^2$.
4. $1 - 196y^2$.
5. $x^4 - 25y^2$.
6. $9a^2 - 4b^2$.
7. $64a^2b^2 - 1$.
8. $25x^2 - 4y^2$.
9. $49m^2 - 16x^4y^2$.
10. $121b^2x^2y^2 - 225z^2$.
11. $a^2 - (b - c)^2$.
12. $(a - b)^2 - m^2$.
13. $(a - b)^2 - (c + d)^2$.
14. $1 - (a - b)^2$.
15. $(a - b)^2 - 1$.
16. $25 - (a^2 - b)^2$.
17. $(am + c^2)^2 - 36$.
18. $a^4 - (c^2 - d^2)^2$.
19. $(a^2 - bc)^2 - (a - d)^2$.
20. $(m - n^2)^2 - (m^2 + n^2)^2$.

EXERCISE IV

Factor:

1. $a^2 - 2ab + b^2 - 1$.
2. $1 - 4a + 4a^2 - x^2$.
3. $x^2 + y^2 + 2xy - 9b^2$.
4. $1 - x^2 + 2xy - y^2$.
5. $9 - m^2 + 2mn - n^2$.
6. $x^2 + 9 - 6x - 25a^2$.
7. $16x^2 + 8x - 100a^2 + 1$.
8. $1 - 6a - 49b^2 + 9a^2$.
9. $9a^2 - 9b^2 + 42b - 49$.
10. $4x^2 - 16a^2b^2 - 20xy + 25y^2$.
11. $a^2 + 2ab + b^2 - c^2 + 2cd - d^2$.
12. $c^2 - 2cd + d^2 - a^2 + 2ab - b^2$.
13. $c^2 + 2cd + d^2 - a^2 - 2ab - b^2$.
14. $4x^2 - 4xy + y^2 - a^2 + 2ab - b^2$.
15. $x^4 - 6x^2y + 9y^2 - 9a^2 + 30ad^2 - 25d^4$.
16. $4m^2 + 20mn + 25n^2 - 9c^2 - 12cd - 4d^2$.
17. $x^4 + 2x^2y^2 + y^4 - 49c^2 + 14c - 1$.
18. $1 + 4x + 4x^2 - x^4 - 4x^2y - 4y^2$.

EXERCISE V

Factor:

1. $x^2 + 14x + 49.$
2. $x^2 - 14x + 49.$
3. $x^6 - 4x^3 + 4.$
4. $9x^2 - 24x + 16.$
5. $4x^2 - 20xy + 25y^2.$
6. $25a^2 + 70ab + 49b^2.$
7. $36a^4 - 84a^2c + 49c^2.$
8. $289x^2 + 136xy + 16y^2.$
9. $81x^2 - 234xy + 169y^2.$
10. $484x^4 - 44x^2y^2 + y^4.$
11. $256x^2y^2 - 96cxy + 9c^2.$
12. $49x^2y^2 + 14d^2xy + d^4.$
13. $144c^2 - 24c + 1.$
14. $1 + 28m^2 + 196m^4.$
15. $49x^4 - 28x^2 + 4.$
16. $81m^4 + 144m^3 + 64m^2.$
17. $25m^4 + 130m^2n + 169n^2.$
18. $(a + x)^2 - 2(a + x) + 1.$
19. $1 - 4(a + b) + 4(a + b)^2.$

EXERCISE VI

Factor:

1. $x^2 + 5x + 6.$
2. $x^2 - 2x - 3.$
3. $x^2 - 4x - 5.$
4. $x^2 - 5x + 4.$
5. $x^2 - 3x - 10.$
6. $x^2 - 3x - 28.$
7. $x^2 + 5x - 24.$
8. $x^2 - x - 20.$
9. $x^2 - 8x + 7.$
10. $x^2 - 5x - 36.$
11. $a^2 + 2a - 35.$
12. $a^2 - 7a - 18.$
13. $a^4 - 8a^2 - 33.$
14. $a^4 + 12a^2 + 11.$
15. $a^2 - 11a - 42.$
16. $a^4b^4 - 9a^2b^2 - 136.$
17. $a^2 - 26a + 133.$
18. $a^4x^4 - 19a^2x^2 - 92.$
19. $x^2 + bx + cx + bc.$
20. $x^2 - bx + cx - bc.$
21. $x^2 - bx - cx + bc.$
22. $x^2 + bx - cx - bc.$
23. $x^2 + (a - d)x - ad.$
24. $x^2 + (a + d)x + ad.$

EXERCISE VII

Factor:

1. $6x^2+7x+2$.
2. $4x^2+8x+3$.
3. $2x^2-3x+1$.
4. $3x^2-8x+4$.
5. $6x^2-x-1$.
6. $2x^2+3x-2$.
7. $3x^2-8x+5$.
8. $2x^2+x-21$.
9. $3x^2+11x-20$.
10. $8x^2+10x-12$.
11. $6x^2-7x-20$.
12. $3x^2-29x+40$.
13. $16x^2-82x-33$.
14. $26x^2-141x-11$.
15. $34x^2+131x-99$.
16. $30x^2-9x-3$.
17. $28x^2+23x-15$.
18. $18x^2-x-39$.
19. $10x^2-59x+85$.
20. $95x^2-138x+7$.
21. $16x^2-211x+39$.
22. $24x^2-54x-105$.
23. $84x^2+148x-112$.
24. $15x^2-101x-28$.
25. $6x^2-61x-55$.
26. $60x^2-147x-156$.
27. $6x^2-77x+92$.
28. $80x^2-28x-52$.
29. $10x^2+9x-91$.
30. $84x^2-158x+70$.
31. $abx^2-x(b+ac)+c$.
32. $ax^2+x(1+ab)+b$.
33. $abx^2+x(ad-bc)-cd$.
34. $2a^2x^2+x(2ac-ab)-bc$.
35. $abx^2+x(ac-b^2)-bc$.
36. $abx^2-x(a^2+bc)+ac$.
37. $2abx^2-x(bc+4a^2)+2ac$.
38. $mnx^2-x(an-3bm)-3ab$.
39. $2ax^2-x(3ad+4c)+6cd$.
40. $7ax^2-x(a-14d)-2d$.
41. $8mnx^2-x(12an-10bm)-15ab$.

EXERCISE VIII

Factor:

1. $x^3 - y^3.$

2. $x^3 + y^3.$

3. $8 + a^3.$

4. $a^3 - 8.$

5. $x^6 + 64.$

6. $125 a^3 - 8 b^3.$

7. $8 b^3 - 125 a^3.$

8. $343 c^3 - 1.$

9. $1 + 1000 b^3.$

10. $512 d^3 - x^{15}.$

11. $8 x^3 + y^3.$

12. $y^3 - 27 z^3.$

13. $1 - 125 a^3.$

14. $216 x^3 y^3 + 27 z^9.$

15. $a^3 b^6 - c^{12}.$

16. $64 c^3 - b^3.$

17. $1 - 343 b^6.$

18. $729 m^3 - 1.$

19. $1000 a^3 + 1.$

20. $m^3 - (a - b)^3.$

21. $(a + b)^3 + c^3.$

22. $(m - n)^3 + (a + b)^3$

23. $8 (a^2 - b^2)^3 - 1.$

24. $27 (a - c)^3 + 64 (b - d)^3.$

25. $343 (c^2 + d^2)^3 - 1.$

26. $(6 c^3 - a^3)^3 - (a^2 + b^2)^3.$

27. $(s + a)^3 - (s - c)^3.$

28. $1728 (x + y)^3 - 343 (x - y)^3.$

29. $729 (a - b)^3 + 125 (a + b)^3.$

30. $512 (a + b + c)^3 - 1331 (a - b - c)^3.$

31. $x^3 (a - b + c)^3 + a^3 (x - y + z)^3.$

32. $x^6 (a + b + c)^3 - b^3 (x + y + z)^3.$

33. $a^6 b^3 (x^2 + y - z)^3 + a^3 b^9 (x - y - z^2)^3.$

34. $a^3 (x + y - z)^3 - b^3 (y + z - x)^3.$

35. $343 a^6 (s + a - b)^3 + 1000 b^6 (a + b + s)^3.$

EXERCISE IX

Factor:

1. $mx + am + nx + an$.
2. $mx - am - nx + an$.
3. $2bx^2 - abx + 4cx - 2ac$.
4. $m^2n - 3abn - 2m^2p + 6abp$.
5. $x^2 + ax + bx + ab$.
6. $x^2 - ax - bx + ab$.
7. $6bx - 15ab - 4dx + 10ad$.
8. $dmnx - ac^2dx + mnrs - ac^2rs$.
9. $-2an + 3ap + 2bn - 3bp$.
10. $6ce - 9de + 4ac - 6ad$.
11. $a^4b^2 - 2a^2b^4 + a^3b - 2ab^3$.
12. $2bcm - 4ab^2c + 7amn - 14a^2bn$.
13. $2m^2n - 2a^2b - 3cm^2n + 3a^2bc$.
14. $-a^2bx^2 - 4bc - 2a^2x^2y - 8cy$.
15. $8as^2 + 10art + 12bs^2 + 15brt$
16. $-mnx - 2mn + p^2x + 2p^2$.
17. $14a^2ce^2f + 35b^2de^2f + 6a^2cxy + 15b^2dxy$.
18. $10ac + bc - 110ad - 11bd$.
19. $rs + a^2n - 3d^2rs - 3a^2d^2n$.
20. $8acxy - 14a^2xz + 21acdz - 12c^2dy$.
21. $6a^3 - 8a^2b^2 - 15abc + 20b^3c$.
22. $6x^3 - 33acx - 8cx^2 + 44ac^2$.

EXERCISE X

Factor:

1. $x^2 - 22x + 121.$

2. $4x^2 - 49a^4.$

3. $x^2 + 28xy + 196y^2.$

4. $343x^3 - 1.$

5. $x^4 - 4x^2 - 60.$

6. $9a^2b^2 - y^2.$

7. $1 + a^2 + b^2 - 2a - 2b + 2ab.$

8. $1 - 3a + 3a^2 - a^3.$

9. $3x^2 + 10xy + 3y^2.$

10. $m^4 + m^2n^2 + n^4.$

11. $81a^4 - 16b^4.$

12. $a^2 - b^2 + 2bc - c^2.$

13. $(m - n)^2 + 2x(m - n) + x^2.$

14. $a^4x^4 + a^2x^2 + 1.$

15. $1 - x^2 - y^2 - 2xy.$

16. $x^2 - 6ax - 9b^2 - 18ab.$

17. $x^3 + x^2y + xy^2 + y^3.$

18. $(2a + b)^2 - (2b + a)^2.$

19. $y^2 - y - 6.$

20. $6x^2 + 13x - 5.$

21. $(27y^3)^2 - 2(27y^3)(8b^3) + (8b^3)^2.$

22. $1 - a^2x^2 - b^2y^2 + 2abxy.$

23. $a^2 + b^2 + 2ab - 4a^2b^2.$

24. $1 - 18x - 63x^2.$

25. $4x^2 - 5x + 1.$

26. $4a^2b^2 - (a^2 + b^2 - c^2)^2.$

27. $nx - x + y - ny.$

28. $x^4 - 16y^4.$

29. $a^4b^2 - 6a^3b + 9a^2.$

30. $28x^4y + 64x^3y - 60x^2y.$

31. $x^4 + y^4 - 18x^2y^2.$

32. $3x^3 + 2x^2 - 2x - 1.$

33. $(7x + 4y)^2 - (2x - y)^2.$

34. $250(a - b)^3 + 2.$

35. $a^2b^2 - a^2 - b^2 + 1.$

36. $x^2y^2z^2 - x^2z - y^2z + 1.$

37. $3x^2 - 6x + 9.$

38. $2x^3 + 5x^2 - 12x.$

39. $27a^3 + 125b^3.$

40. $x^2y + 3xy^2 - 3x^3 - y^3.$

41. $56 + x - x^2.$

42. $5x^3y^2 + 5x^2yz - 60xz^2.$

43. $x^3 + x^2 + x + 1.$

44. $x^2 - 18x + 32.$

45. $x^2 - 2ax - 2bx + 4ab.$

46. $x^4y^2 + 2x^2y^2z^2 + y^2z^4.$

47. $a^2x^2 - 2acxz - b^2y^2 + c^2z^2.$

48. $a^6 - b^6.$

49. $a^3 + b^3 + a + b.$

50. $a^3 - 2a^2b + 2ab^2 - b^3.$

51. $(5x - 2)^2 - (x - 4)^2.$

52. $1 + bx - (a^2 + ab)x^2.$

53. $a^2x^2 - a^2y^2 - b^2x^2 + b^2y^2.$

54. $4(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2.$

55. $a^3 - 5a^2b + 4b^3.$

56. $64a^6 - 1.$

57. $y^2 - c^2 + 2cx + x^2.$

58. $bc(b - c) + ca(c - a) + ab(a - b).$

59. $x^3 - 2x^2y + x^2 - 4x + 8y - 4.$

60. $a^3 - a^2b + ab^2 - b^3.$

61. $a^2(b - c) + b^2(c - a) + c^2(a - b).$

62. $x^{12} - y^{12}.$

63. $x^{12} + y^{12}.$

64. $110 - x - x^2.$

65. $acx^2 - bcx + adx - bd.$

66. $a^4b + 8ac^3bm^6.$

67. $4c^3x^2 + 4c^2xy + cy^2.$

68. $9x^4 - 40x^2y^2 + 16y^4.$

69. $x^5 + y^5.$

70. $a^4 - a^3y + az^3 - yz^3.$

71. $x^4 - (x - 6)^2.$

72. $m^6 - 64n^6.$

73. $a^9 - x^6.$

74. $8(x + y)^3 - (2x - y)^3.$

75. $x^3 - 7x^2 + 14x - 8.$

76. $3x^2 + x(3a + b) + ab.$

77. $x^4 - 5x^2y^2 + 4y^4.$

78. $(x^3 + x - 1)^2 - (x^3 - x - 1)^2.$

79. $x^2 + 3x^3 - x^4 - 3x.$

80. $a^{10} + b^{10}.$

81. $x^7 + y^7.$

82. $x^3 - 3x - 2.$

83. $(a-2b)^2-9-3(a-2b+3)$.

84. $a^2-b^2+x^2-y^2+2(ax-by)$.

85. $mx^3+anx^2-b^2nx-ab^2m+nx^3+amx^2-b^2mx-ab^2n$.

86. $x^4+2x^3+2x^2+2x+1$.

87. $2x^4+x^3+4x^2+6x+2$.

88. $m^4-m^3z-m^2z-mz-z-1$.

89. $2x^2+3xy+y^2-2x-y$.

90. $9(a^2-ac)^2-6ac^2(a-c)+c^4$.

91. $(a-b)(2x^2-2xy)+(b-a)(2xy-2y^2)$.

92. $(x-1)(x-2)(x-3)+(x-1)(x-2)-(x-1)$.

93. $x^3-2x-21$.

94. $x^2-2xy-3y^2-x^3+2x^2y+3xy^2$.

95. $(x+y)^3+x+y$.

96. $(m+n-3)^2-3(m+n-3)-4$.

97. $x^6-y^6-(x-y)^2$.

98. $(a^2-b^2)+xy(3b+a)-2x^2y^2$.

99. $m^3-3m^2n+3mn^2-n^3-m+n$.

100. $x^3+y^3+z^3-3xyz$.

101. $(x-y)^3+1-3(x-y+1)$.

102. $x^4+x^3+x^2y^2+x^2+y^2+xy^2$.

103. $(x+y)^4-x^4-y^4$.

104. $(a^3-b^3)-(a^2-b^2)-(a-b)^2$.

105. $a^4+b^4+c^4-2a^2b^2-2a^2c^2+2b^2c^2$.

106. $4b^4c^4-b^4-2b^2c^2-c^4$.

107. $x^4+x^3+x^2y^2-y^3+y^4$.

EXERCISE XI

Find, by factoring, the H. C. F. of the following expressions:

1. $ab + ab,\ b^2 + b.$
2. $15x - 9,\ 6 - 10x.$
3. $ax^2 - 2axy,\ 2ax^2 - axy.$
4. $am - an + bm - bn,\ am - an.$
5. $(x+1)^2,\ x^2 + x.$
6. $a^2 - 4a + 4,\ 3ab - 6b.$
7. $a^2 - 6a + 9,\ ab - 3a - 3b + 9.$
8. $x^2 - 1,\ x^2 - x.$
9. $x^4 - y^4,\ x^2 + y^2.$
10. $a^3 + 1,\ a^2 - 1.$
11. $25a^2 - 9(b-1)^2,\ 6b - 10a - 6.$
12. $x^2 + 3x + 2,\ x^2 + x - 2.$
13. $x^2 - 9x + 20,\ x^2 - 16.$
14. $4a^2 - 5ab - 6b^2,\ 8a^2 + 2ab - 3b^2.$
15. $x^4 + x^2y^2 + y^4,\ x^3 + y^3.$
16. $x^2 - 8x + 12,\ x^2 - 7x + 6,\ x^3 - 216.$
17. $3(x-1)^3,\ x^4 - 1,\ x^3 + x^2 - 2.$
18. $x^3 - y^3,\ x^4 + x^2y^2 + y^4,\ x^3 + 4x^2y + 4xy^2 + 3y^3.$
19. $2x^2 + 17x + 21,\ 8x^3 + 27,\ 2x^2 + 5x + 3.$
20. $m^2 - 4m + 3,\ m^2 - 6m + 9,\ m^3 - 9m^2 + 27m - 27.$
21. $a^4 - 16,\ a^3 + 2a^2 + 4a + 8,\ 3a^3 - 2a^2 + 12a - 8.$
22. $m^3 - n^6,\ m^3 - 3m^2n^2 + 3mn^4 - n^6,\ m^4 - mn^6 + m^3n - n^7.$
23. $x^3 - 27,\ x^4 + 9x^2 + 81,\ x^3 + 2x^2 + 6x - 9.$
24. $3x^3 + 2x^2 - 2x - 1,\ x^4 - 1,\ ax + a - bx - b.$
25. $(a-b)^3 + 1,\ (a-b)^2 - 1,\ (a-b)^2 - 2(a-b) - 3.$

EXERCISE XII

Find, by factoring, the L. C. M. of the following expressions:

1. $2x^2y^3$, $3x^3y^2$, $5x^4y$, $7xy^7$.
2. $4x^3y$, $5x^2y^3$, $6xy^4$, $15y^4$.
3. $4a^5b$, $6a^4b^2$, $18a^3b^3$, $36a^2b^4$.
4. $5a^3b^2c^2$, $7a^3b^7c^4$, $91a^4b^3c^5$, $65a^7b^4c^6$.
5. $x^2 - xy$, $7x^2 - 7y^2$.
6. $(a+b)$, (a^2-b^2), $a^2+2ab+b^2$.
7. x^2-y^2, x^3+y^3, x^3-y^3.
8. a^4-b^4, $a^4+2a^2b^2+b^4$, a^4+b^4.
9. $(m-n)^3$, m^3-n^3, m^4-n^4.
10. x^5-y^5, x^3+y^3, x^6-y^6.
11. m^2-2m-3, m^3-27.
12. $x^2-2xy+y^2-1$, $(x-y)^3-1$.
13. x^3+64, x^2+x-12.
14. $x^{12}-y^{12}$, x^6+y^6, x^4+y^4.
15. $4x^2+4x+4$, $6x^3-6$.
16. $3(a-b)^3$, $27a^3-27b^3$.
17. $3x^2+14x+8$, $27x^3+8$.
18. $(a-b)^2-c^2$, $(a-b+c)^2$.
19. x^3-5x+2, x^4-16.
20. x^3+x-2, $(x^2+2)^2-x^2$.
21. $(x-y)^2-z^2$, $(x-z)^2-y^2$, $(y-z)^2-x^2$.
22. $6x^3(x^3-y^3)$, $3xy(x-y)^3$, $2y^3(x^6-y^6)$.

EXERCISE XIII

Find the algebraic sum of the following fractions:

1. $\frac{3}{m}-\frac{x}{m}.$

2. $\frac{3a}{7}-\frac{a}{7}.$

3. $\frac{m}{x^2}=\frac{n}{x}.$

4. $\frac{a}{3x}+\frac{b}{4x}.$

5. $\frac{1}{6a}+\frac{1}{9b}.$

6. $\frac{a}{x^2}-\frac{b}{xy}.$

7. $7+\frac{1}{a}.$

8. $\frac{a-x}{x}+1.$

9. $1-\frac{b}{a+b}.$

10. $\frac{5a-17b}{16}+\frac{7b-2a}{12}-\frac{a-4b}{8}.$

11. $\frac{2x-5y-3}{15}-\frac{3x-8y+45}{25}+2.$

12. $\frac{a+4b}{10}-\frac{2(a-3b)}{15}-\frac{11(a+b)}{20}.$

13. $\frac{3(x-y)}{4}-\frac{5(2x-3y)}{6}+\frac{7(x-2y)}{8}.$

14. $\frac{5a+16b-14}{33(a+1)}+\frac{a-6b+7}{36(a+1)}-\frac{3a+14b-15}{44(a+1)}.$

15. $\frac{4a+7c}{6bc}-\frac{3}{c}+\frac{4b-3a}{2ab}+\frac{3a-2c}{ac}.$

16. $\frac{5x+3y}{3x^2}+\frac{y+2z}{4yz}-\frac{3x+4y}{6xy}-\frac{x+3z}{4xz}-\frac{1}{4x}.$

17. $\frac{a^2+4bc}{8ac}-\frac{b^2-3ac}{6ab}-\frac{ab+4c^2}{4bc}-\frac{b}{3a}+\frac{c}{2b}.$

18. $\frac{2}{5a}+\frac{2(6a-b)}{15ab}-\frac{1}{3b}-\frac{a+b}{10a^2}-\frac{3a^2-b^2}{10a^2b}.$

19. $\frac{a-2b}{4ab}+\frac{3a-4b}{6a^2}-\frac{1}{2b}-\frac{6a-5b}{20b^2}+\frac{a^2+2b^2}{3a^2b}.$

20. $\frac{4(ab+xy)}{15\,bx}+\frac{ay-8\,b^2}{6\,ab}-\frac{a^2-5\,bx}{6\,ax}+\frac{b}{2\,a}-\frac{y}{3\,b}-\frac{a}{10\,x}$.

21. $\frac{(1+x)^2}{x^2}-\frac{(1+y)^2}{y^2}+\frac{2\,x-2\,y+1}{xy}+\frac{2\,x+y}{xy^2}$.

22. $\frac{2}{3}+\frac{1}{a-1}$.

23. $\frac{a}{b}-\frac{a+m}{b+m}$.

24. $\frac{x}{y}=\frac{y}{x}=\frac{x-y}{x+y}$.

25. $\frac{a}{1-x}-\frac{a}{1+x}$.

26. $\frac{1}{a-b}-\frac{1}{a+b}$.

27. $\frac{a}{a+b}-\frac{b}{a-b}$.

28. $\frac{3}{x-11}-\frac{2}{x-7}$.

29. $\frac{3(x+2)}{x+5}-\frac{5(x-2)}{2\,x+1}$.

30. $\frac{x-13}{10(x-3)}-\frac{x-18}{14(x-2)}$.

31. $\frac{5}{3\,x-3}-\frac{3}{2\,x-2}$.

32. $\frac{2\,x-13}{10\,x+10}-\frac{3\,x-16}{15\,x+45}$.

33. $\frac{3}{2\,x-4}-\frac{5}{6\,x-12}-\frac{2}{3\,x+6}$.

34. $\frac{1}{y}+\frac{x-y}{x^2+xy}-\frac{2}{x+y}$.

35. $\frac{x-y}{xy}-\frac{x-a}{ax+bx}+\frac{y-b}{ay+by}$.

36. $\frac{5}{4(a+b)}+\frac{3}{a-b}-\frac{13\,a+7\,b}{4(a^2-b^2)}$.

37. $\frac{5}{x-2}+\frac{7}{x-3}-\frac{x-4}{x^2-5\,x+6}$.

38. $\frac{21\,x+13}{12\,x+24}-\frac{5\,x}{3\,x-3}+\frac{16\,x-3}{4\,x^2+4\,x-8}$.

EXERCISE XIV

Find the algebraic sum of the following fractions:

1. $\frac{1}{x^2-1}-\frac{1}{x+1}-\frac{1}{1-x}.$

2. $\frac{x^2}{x^2-y^2}-\frac{1}{y-x}+\frac{x^2}{y^2-x^2}.$

3. $\frac{a}{a-b}-\frac{2a}{a+b}+\frac{3ab}{b^2-a^2}.$

4. $\frac{3}{x-3y}-\frac{2}{3y+x}+\frac{15y}{9y^2-x^2}.$

5. $\frac{1}{(x-1)(x-2)}-\frac{1}{(2-x)(3-x)}-\frac{1}{(1-x)(x-3)}.$

6. $\frac{1}{(a-b)(b-c)}-\frac{1}{(b-a)(a-c)}+\frac{1}{(c-b)(c-a)}.$

7. $\frac{c+a}{(a-b)(b-c)}+\frac{b+c}{(a-b)(a-c)}+\frac{a+b}{(c-b)(c-a)}.$

8. $\frac{a^3}{(a-b)(a-c)}+\frac{b^3}{(b-a)(b-c)}+\frac{c^3}{(c-a)(c-b)}.$

9. $\frac{(a-m)(b-m)}{(a-c)(b-c)}+\frac{(b-m)(c-m)}{(b-a)(c-a)}+\frac{(c-m)(a-m)}{(c-b)(a-b)}.$

10. $$\frac{a}{(a-b)(a-c)(a-d)}+\frac{b}{(b-a)(b-c)(b-d)}$$
$$+\frac{c}{(c-a)(c-b)(c-d)}+\frac{d}{(d-a)(d-b)(d-c)}.$$

11. $\frac{6}{a}-\frac{4}{b+a}+\frac{4}{b-a}+\frac{1}{a-2b}-\frac{1}{-2b-a}.$

12. $\frac{1}{a+2x}+\frac{1}{2x-a}+\frac{4x}{a^2+4x^2}-\frac{32x^3}{a^4+16x^4}.$

EXERCISE XV

Find the product of:

1. $\frac{x^2-2xy+y^2}{x^2+y^2}\cdot\frac{x^4-y^4}{x+y}$.

2. $\frac{3a^3}{a^3+b^3}\cdot\frac{a^2-ab+b^2}{12a^4}$.

3. $\frac{2x+14}{a-3}\cdot\frac{a^2-9}{3x+21}$.

4. $\frac{(x-y)^4}{x^2+y^2}\cdot\frac{x^4-y^4}{(x-y)^5}$.

5. $\frac{a^3+27}{a^2-16}\cdot\frac{a+4}{a^2-3a+9}$.

6. $\frac{a^3-64}{a^3+64}\cdot\frac{a^2-16}{a^2+16}$.

7. $\frac{ab+b^2}{a^3-b^3}\cdot\frac{a^2-b^2}{a^3+b^3}$.

8. $\frac{x^3-1}{x^2+x}\cdot\frac{x^2-1}{x^4+x^2+1}$.

9. $\frac{m^2-7m+12}{m^2-3m+2}\cdot\frac{m^2-6m+8}{m^2-8m+16}$.

10. $\frac{2a^3+a^2b-2ab^2-b^3}{a^3+2a^2b+ab^2+2b^3}\cdot\frac{a+2b}{2a+b}$.

11. $\frac{m^3+2m^2+2m+1}{m^3+1}\cdot\frac{m^2+2m+1}{m^3-1}$.

12. $\frac{x^4-y^4}{x^5+x^4y+xy^4+y^5}\cdot\frac{x^4+y^4}{x^2+y^2}$.

13. $\frac{x^2-x-6}{x^2y+xy-2y}\cdot\frac{x^3+3x^2-4x}{x^2+x-12}$.

14. $\frac{2x^2-5x+3}{3x^2-8x+4}\cdot\frac{3x^2-11x+6}{2x^2+5x-12}\cdot\frac{8-2x-x^2}{x^2-4x+3}$.

15. $\left(\frac{9a^2}{b^2}-1\right)\left(\frac{3a}{3a-b}-1\right)\left(1-\frac{3a}{3a+b}\right)$.

16. $\frac{(a^3-b^3)}{a^2-b^2}\cdot\frac{(a-b)^3}{a^2(a+b)}\cdot\frac{a^3+b^3}{a^4+a^2b^2+b^4}$.

17. $\frac{2x^2+2ax-8x-8a}{x^2-ax+2x-2a}\cdot\frac{2x^2-2ax-3x+3a}{x^2+ax-4x-4a}\cdot\frac{x+2}{2x-3}$.

EXERCISE XVI

Find the quotient of:

1. $\frac{3a\cancel{x}}{8} \div \frac{5a}{12\cancel{x}y}$.

2. $\frac{55a}{24x} \div \frac{15a^2x^2}{4}$.

3. $\frac{36x^3y^2}{27a^2b^3} \div \frac{24x^2y^3}{81a^3b}$.

4. $\frac{40a^7b^5c^8}{22m^3x^4z^5} \div \frac{35a^6b^6c^8}{88m^6xz^7}$.

5. $\frac{9y(a-4b)}{22a^2x(a+b)} \div \frac{6y^2(4b-a)}{55ax^2(a-b)}$.

6. $\frac{b-a}{b} \div (a-b)$.

7. $\left(1+\frac{a}{b}\right) \div \left(1+\frac{b}{a}\right)$.

8. $(\frac{1}{3}a+\frac{1}{2}b) \div (\frac{1}{4}a+\frac{1}{8}b)$.

9. $\left[\frac{a^3+b^3}{a^4+b^4} \div \frac{a^2+ab+b^2}{a^8-b^8}\right] \div \frac{a^4-b^4}{a+b}$.

10. $\left[\frac{a^3-b^3}{a^2+b^2} \div \frac{(a^2-b^2)^2}{a^4-b^4}\right] \div \frac{a^4+b^4}{a^8-b^8}$.

11. $\left[\left(x-\frac{y^2}{x}\right) \div (x-y)\right]\frac{x^2}{x+y}$.

12. $\left[\frac{(a+b)^2-c^2}{(b+c)^2-a^2} \div \frac{a^2-(b-c)^2}{c^2-(b-a)^2}\right]\frac{(a+b+c)^2}{(a-b-c)^2}$.

13. $\left[\frac{x^3-2x^2-2x+1}{x^2-2x+1} \div \frac{x^2+x+1}{x^2-1}\right]\frac{x^3-1}{(x+1)^3}$.

14. $\left[\frac{x^2-7x+6}{x^2-3x+2} \div \frac{x^2-9x+20}{x^2-7x+12}\right]\frac{x^2-7x+10}{x^2-9x+18}$.

15. $\left[\frac{(x+y)^2-z^2}{(x+y+z)^2} \cdot \frac{(x+z)^2-y^2}{(x-y+z)^2}\right] \div \frac{y^2-(x-z)^2}{x^2-(y+z)^2}$.

16. $\left[\frac{4x^2-5x+1}{8x^2-2x-1} \div \frac{3x^2-4x+1}{3x-1-2x^2}\right] \div \frac{1-3x+2x^2}{5x-1-6x^2}$.

17. $\left[\frac{2m^2-m-6}{m^2-5m+6} \cdot \frac{2m^2-m-3}{6m^2-11m+3}\right] \div \frac{2m^2-m-3}{3m^2-10m+3}$.

EXERCISE XVII

Simplify the following fractions:

1. $\dfrac{5x^2+4x-1}{20x^2+21x-5}$.

2. $\dfrac{12x^2+24x-15}{4x^2+12x+5}$.

3. $\dfrac{\dfrac{x^2+y^2}{y}-x}{\dfrac{1}{y}-\dfrac{1}{x}}\cdot\dfrac{x^2-y^2}{x^3+y^3}$.

4. $\dfrac{3x^3+6x^2-3x-6}{x^3+3x^2+2x}$.

5. $1-\dfrac{\dfrac{a^2+3a+2}{a^2+2a+1}}{\dfrac{a^2+7a+12}{a^2+5a+4}}$.

6. $\dfrac{x^3-a^3}{x^3+a^3}+\dfrac{(x-a)^2}{x^2-a^2}$.

7. $\dfrac{x^3-8y^3}{x^2-y^2}\cdot\dfrac{x+y}{x^2+2xy+y^2}$.

8. $\dfrac{xy^2+y^3}{x^2+xy+y^2}\div\dfrac{x^2y^2-x^4}{y^3-x^3}$.

9. $\dfrac{\dfrac{a^2+b^2}{b}-a}{\dfrac{a}{b}-1}\cdot\dfrac{a^2-b^2}{a^3+b^3}$.

10. $\dfrac{a^4-x^4}{a^2-2ax+x^2}\div\dfrac{a^2x+x^3}{a^3-x^3}$.

11. $\dfrac{x^2-9x+20}{x^2-6x}\cdot\dfrac{x^2-13x+42}{x^2-5x}$.

12. $\left(a+\dfrac{b-a}{1+ab}\right)\dfrac{a}{b}\div\left(1-a\dfrac{b-a}{1+ab}\right)$.

13. $\left(\dfrac{x}{1+x}+\dfrac{1-x}{x}\right)\div\left(\dfrac{x}{1+x}-\dfrac{1-x}{x}\right)$.

14. $\left(\dfrac{2}{x}-\dfrac{1}{a+x}+\dfrac{1}{a-x}\right)\div\left(\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}\right)$.

15. $\dfrac{xy}{x^2+y^2}\left(\dfrac{x+y}{x-y}+\dfrac{x^3+y^3}{x^3-y^3}\right)\div\left(\dfrac{x+y}{x-y}-\dfrac{x^3+y^3}{x^3-y^3}\right)$.

16. $\dfrac{3}{x+1}-\dfrac{2x-1}{x^2+\dfrac{x}{2}+\dfrac{1}{2}}.$

17. $\dfrac{3x^2+2x-1}{x^3+x^2-x-1}.$

18. $\dfrac{2m-3+\dfrac{1}{m}}{\dfrac{2m-1}{m}}.$

19. $\dfrac{\dfrac{a^2+b^2}{b}-a}{\dfrac{1}{b}-\dfrac{1}{a}}\cdot\dfrac{a^2-b^2}{a^2+b^2}.$

20. $\dfrac{a^4-x^4}{a^2-2ax+x^2}\div\dfrac{a^2-x^2}{(a-x)^3}.$

21. $\dfrac{2x^2+5x+2}{x^2-4}\div\dfrac{2x^2+9x+4}{x+4}.$

22. $\dfrac{\dfrac{1}{x-y}-\dfrac{x}{x^2-y^2}}{\dfrac{x}{xy+y^2}-\dfrac{y}{x^2+xy}}\cdot\dfrac{x^2-2xy+y^2}{xy}.$

23. $\left(\dfrac{x^4-y^4}{x^2-y^2}\div\dfrac{x+y}{x^2-xy}\right)\div\left(\dfrac{x^2+y^2}{x-y}\div\dfrac{x+y}{xy-y^2}\right).$

24. $\left(1-\dfrac{ab}{a^2-ab+b^2}\right)\left(1-\dfrac{ab}{a^2+2ab+b^2}\right)\div\dfrac{a^3-b^3}{a^3+b^3}.$

25. $\dfrac{a^2-bc}{(a-b)(a-c)}+\dfrac{b^2+ca}{(b+c)(b-a)}+\dfrac{c^2+ab}{(c-a)(c+b)}.$

26. $\dfrac{x^2-yz}{(x+y)(x+z)}+\dfrac{y^2-zx}{(y+z)(y+x)}+\dfrac{z^2-xy}{(z+x)(z+y)}.$

27. $\dfrac{1}{x^2-\dfrac{x^3+1}{x+\dfrac{1}{x-1}}}.$

28. $\dfrac{a}{1-\dfrac{a}{1-\dfrac{1}{a-1}}}.$

29. $\dfrac{\dfrac{a-b}{1+ab}+\dfrac{b-c}{1+bc}}{1-\dfrac{(a-b)(b-c)}{(1+ab)(1+bc)}}.$

30. $\dfrac{\left(a^2+\dfrac{b^4}{a^2+b^2}\right)(a^2-b^2)}{1-\dfrac{b}{a+b}-\dfrac{1}{2}\left(1-\dfrac{a-b}{a+b}\right)}.$

31. $(a^2+b^2)\left\{\dfrac{\dfrac{b^4}{b^2-a^2}-a^2}{\dfrac{a}{a+b}+\dfrac{b}{a-b}}\right\}.$

32. $\dfrac{\dfrac{1-x^2}{1+y}\left(\dfrac{x}{1+x}-1\right)}{1-\left(\dfrac{1}{1-y}-\dfrac{x^2+y^2-x+y}{1-y^2}\right)}.$

33. $\left(\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}\right)\div\left(\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}\right).$

34. $\dfrac{\dfrac{1}{a-x}-\dfrac{1}{a-y}+\dfrac{x}{(a-x)^2}-\dfrac{y}{(a-y)^2}}{\dfrac{1}{(a-y)(a-x)^2}-\dfrac{1}{(a-x)(a-y)^2}}.$

35. $\dfrac{y-z}{y+z}+\dfrac{z-x}{z+x}+\dfrac{x-y}{x+y}+\dfrac{(y-z)(z-x)(x-y)}{(y+z)(z+x)(x+y)}.$

36. $\dfrac{a^2-bc}{(a+b)(a+c)}+\dfrac{b^2-ca}{(b+c)(b+a)}+\dfrac{c^2-ab}{(c+a)(c+b)}.$

37. If $\dfrac{x}{y+z}=a$, $\dfrac{y}{x+z}=b$, $\dfrac{z}{x+y}=c$, find the value of $\dfrac{a}{1+a}+\dfrac{b}{1+b}+\dfrac{c}{1+c}.$

EXERCISE XVIII

Solve the following equations :

1. $\frac{1}{x+2}+\frac{7}{3(x+2)}=\frac{2}{3}.$

2. $\frac{5}{x+1}-\frac{3}{2x+2}=\frac{5}{2}.$

3. $\frac{9}{2x+2}-\frac{7}{3x+3}=\frac{13}{12}.$

4. $\frac{1}{2-3x}-\frac{1}{4-6x}=\frac{1}{6}.$

5. $\frac{7}{x+1}-\frac{11}{2x+2}=\frac{5}{2}.$

6. $\frac{4}{2x+2}+\frac{31}{3x+3}=\frac{1}{6}.$

7. $\frac{13}{x-2}+\frac{7}{5x-10}=4.$

8. $\frac{15}{3-2x}-\frac{2}{6-4x}=14.$

9. $\frac{1}{3(x-7)}+\frac{1}{6}=\frac{1}{2x-14}.$

10. $\frac{120}{144-x^2}=\frac{x-10}{12+x}+\frac{x+10}{12-x}.$

11. $\frac{5x}{x-3}+\frac{7+4x}{4x-7}=1-\frac{6-5x}{x-3}.$

12. $\frac{3x^2+4x+5}{6x^2+7x+8}=\frac{1}{2}.$

13. $\frac{x-9}{x-12}+\frac{x-4}{x-7}=2.$

14. $\frac{5x^2+7x+4}{15x^2+x-6}=\frac{3x^2+6x+7}{9x^2+6x+3}.$

15. $\frac{7x-2}{5x+3}-\frac{2x+5}{3x+9}=\frac{11x+3}{15x+9}.$

16. $\frac{5x+2}{4x+3}-\frac{3x+1}{6x+2}=\frac{3x+2}{4x-6}.$

17. $\frac{1}{2x+1}-\frac{11}{12}=\frac{x}{2x+1}-\frac{4}{3(2x+1)}.$

18. $\frac{5x+4}{x-1}-\frac{3(x-7)}{5(x-1)}=6-\frac{9}{5(x-1)}.$

19. $\frac{x+3}{4}+\frac{2x-1}{3x-12}-\frac{7x+5}{8x-32}=\frac{x^2-25}{4x-16}.$

20. $\frac{13x+10}{28x-32}+\frac{2(10x+1)}{49x-56}-\frac{7-11x}{35x-40}=3.$

21. $\frac{\frac{1}{2}+x}{\frac{1}{3}-x}+5+\frac{2x^2}{\frac{1}{9}-x^2}=3-\frac{\frac{1}{4}+x}{\frac{1}{3}+x}.$

EXERCISE XIX

Solve the following equations:

1. $\frac{2x+7}{4}-\frac{3x+8}{5x+3}=\frac{4x+3}{8}.$

2. $\frac{6x+1}{15}+\frac{11x-1}{4x+3}=\frac{2x+11}{5}.$

3. $\frac{5x+1}{12}-\frac{3x+2}{5x-8}=\frac{15x-39}{36}$

4. $\frac{2x+3}{10}+\frac{2x-1}{4x+2}=\frac{x+3}{5}.$

5. $\frac{x-3}{x-4}-\frac{x-4}{x-5}=\frac{x-5}{x-6}-\frac{x-6}{x-7}.$

6. $\frac{x+1}{x+2}-\frac{x+2}{x+3}=\frac{x+3}{x+4}-\frac{x+4}{x+5}.$

7. $\frac{21x+13}{3x+1}-\frac{8x+13}{4x+1}=5.$

8. $\frac{9x+2}{3x-1}+\frac{5x+2}{x+1}=8.$

9. $\frac{2x+3}{x-4}+\frac{4x+5}{x-6}=6.$

10. $\frac{x+3}{x+4}-\frac{x+4}{x+5}=\frac{x+5}{x+6}-\frac{x+6}{x+7}.$

11. $\frac{4x+5}{9}-\frac{14x+3}{35x+1}=\frac{16x+3}{36}.$

12. $\frac{x-10}{7}+\frac{13x-2}{10x+7}-\frac{47x-1}{35}+\frac{6x+7}{5}=1.$

13. $\frac{3x+8}{x+1}+\frac{5x+8}{x+2}+\frac{10x+27}{x+3}=18.$

EXERCISE XX

Solve the following equations:

1. $\frac{ax}{b}=1.$

2. $\frac{a}{bx}=\frac{b}{a}.$

3. $\frac{a}{x}-1=\frac{a-1}{x}.$

4. $\frac{x}{m}+\frac{x}{n}=a.$

5. $\frac{m}{ax}+\frac{n}{bx}=c.$

6. $\frac{x}{m}+\frac{x}{n}+\frac{x}{p}=a.$

7. $\frac{ax}{m}+\frac{bx}{n}+\frac{cx}{p}=d.$

8. $\frac{a}{mx}+\frac{b}{nx}+\frac{c}{xp}=d.$

9. $\frac{a}{bcx}+\frac{b}{acx}+\frac{c}{abx}=a^2+b^2+c^2.$

10. $\frac{ab}{cx}+\frac{ac}{bx}+\frac{bc}{ax}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.$

11. $\frac{x+a}{b}+\frac{7}{3}-\frac{2(x-b)}{3a}=4.$

12. $\frac{3(x-4b)}{a}-\frac{5(x-3a)}{4b}=4.$

13. $\frac{ax-1}{b}+\frac{bx-1}{a}=\frac{a^3+b^3}{a^2b^2}.$

14. $\frac{cx+ab}{a}+b=\frac{ax+b^2}{b}+\frac{a^2}{c}.$

15. $\frac{b}{ax}-\frac{b^2}{a^2}-\frac{a}{2bx}=\frac{1}{2}-\frac{a^2}{2b^2}.$

16. $\frac{b+c}{bcx}+\frac{a+c}{acx}-\frac{a+b}{abx}=\frac{2}{c}.$

17. $\frac{ax}{a-b}=1-x.$

18. $\frac{x}{a+b}=\frac{b+x}{a}.$

19. $\frac{a+x}{b+c}-\frac{x}{a}=1.$

20. $\frac{ab-c^2x}{a-b}+\frac{a}{3c}=\frac{cx}{3b}.$

21. $\frac{x}{b+cx}=\frac{a}{c(1+a)}.$

22. $\frac{ax}{a-b}-\frac{bx}{a+b}=a^2+b^2.$

23. $\frac{x-2a+3b}{x+2a-3b}=\frac{5b}{4a-b}.$

24. $\frac{x-a}{a-b}+\frac{x+3b}{a+b}=3.$

25. $\frac{ax+bc}{b+c}-\frac{bx-ac}{a^2+b^2}=\frac{c}{a}.$

26. $\frac{ax^2+bx+1}{bx^2+ax+1}=\frac{a}{b}.$

27. $a\left(\frac{x}{c}-b\right)+\frac{x}{a+b}-a^2=\frac{x-2bc}{a-b}.$

28. $\frac{2bx}{ab+cd}+\frac{cd-x}{a}-\frac{a(x-2ab)}{ab-cd}=a+b.$

29. $\frac{x-ab}{ac}+\frac{x-ac}{bc}+\frac{x-bc}{ab}-3=\frac{bc+a^2}{ac}+\frac{c}{b}.$

30. $\frac{(b-c)x}{bx-c^2}+\frac{cx+b^2}{c(x+b)}=\frac{2b-c}{b}.$

31. $\frac{ax+bc}{bx+ac}-\frac{bx+a(c-1)}{ax-bc}=\frac{a-b}{a}+\frac{(a-b)(x-c)+b}{bx+ac}$

32. $\frac{a}{2x-ab}+\frac{x-ac}{b-c}-\frac{1}{c}=\frac{bx-ac^2}{b(b-c)}-\frac{ac}{b}.$

33. $\frac{3a-x}{b}-\frac{2(x-c)}{3a-2b}+\frac{x+2b}{3a+c}=1-\frac{c}{b}.$

34. $\frac{ax+b}{bx+c}-\frac{ax-c}{bx+2c}=\frac{b^2+c(a+b)}{b(bx+3c)}.$

35. $\frac{abx+1}{a+b}+\frac{acx-1}{a-c}+\frac{bcx-1}{b-c}=\frac{ab+ac+bc}{abc}.$

36. $\frac{ax+b}{x+1}-\frac{b(ax-b)}{bx-a}=\frac{2b(b-a)(2b+a)}{2b(bx-a)+b^2-a^2}.$

37. $\frac{a(x-2c)}{a-b-c}-\frac{b(x-2a)}{c-a-b}+\frac{c(x+2b)}{a+b+c}=x.$

38. $\frac{2a-x}{a-b-c}+\frac{2b-x}{b-a-c}-\frac{2c-x}{c-a-b}=\frac{x}{a+b+c}.$

39. $\dfrac{6a+5b}{6a}-\dfrac{4bx}{3a^2}=1-\dfrac{bx}{a^2+ab}.$

40. $\dfrac{x}{3a+x}+\dfrac{x}{3a-x}=\dfrac{a^2}{9a^2-x^2}.$

41. $\dfrac{x}{p+q}+pqx=p+q+\dfrac{1}{pq}.$

42. $\dfrac{m+nx}{p+qx}=\dfrac{qrx^2}{p(p+qx)}+\dfrac{s-rx}{p}.$

43. $\dfrac{a}{x-a}-\dfrac{b}{x-b}=\dfrac{b^2-a^2}{b^2-bx}.$

44. $\dfrac{ax-b}{ax+b}-\dfrac{bx-a}{bx+a}=\dfrac{a-b}{(ax+b)(bx+a)}.$

45. $\dfrac{b(x-b)}{2a+b+2c}+\dfrac{a(x-a)}{a+2b+2c}+\dfrac{c(x-c)}{2a+2b+c}=\dfrac{x}{2}.$

46. $\dfrac{f+3x}{4f^2+6fg}-\dfrac{2x-f}{6fg-9g^2}=\dfrac{2f+x}{4f^2-9g^2}.$

47. $\dfrac{3bx}{2a^2}-\dfrac{x-b}{a+b}-\dfrac{bx-a^2}{a^2-b^2}+\dfrac{x}{4a}=0.$

48. $\dfrac{a^2+4a}{x^2+x-a^2+a}-\dfrac{a}{x+a}=\dfrac{1}{x-a+1}.$

49. $\dfrac{2(mnx+m^2-n^2)}{n^4-x^2}=\dfrac{m^2+x}{n^2-x}-\dfrac{m^2-x}{n^2+x}.$

50. $\dfrac{x-p}{x-p-1}-\dfrac{x-p-1}{x-p-2}=\dfrac{x-q}{x-q-1}-\dfrac{x-q-1}{x-q-2}.$

51. $\dfrac{3def}{d+e}+\dfrac{d^2e^2}{(d+e)^3}+\dfrac{(2d+e)e^2x}{d(d+e)^2}=3fx+\dfrac{ex}{d}.$

52. $\dfrac{k}{x}+\dfrac{x}{k}+\dfrac{k(x-k)}{x(x+k)}-\dfrac{x(x+k)}{k(x-k)}=\dfrac{kx}{k^2-x^2}-2.$

EXERCISE XXI

In the following equations express each literal quantity in terms of the other literal quantities:

1. $\frac{a}{b}=\frac{c}{d}.$

2. $\frac{x}{r}=\frac{b}{a}.$

3. $V=\frac{S}{T}.$

4. $\frac{1}{A}=\frac{1}{ab}.$

5. $\frac{1}{F}=\frac{2}{a+b}.$

6. $\frac{1}{D_o}+\frac{1}{D_i}=\frac{1}{F}.$

7. $s=\frac{(v_1-v_0)t}{2}.$

8. $\frac{V_1P_1}{T_1}=\frac{V_2P_2}{T_2}.$

9. $tv-tw=t-1.$

10. $C=\frac{5}{9}(F-32).$

11. $Ma-Rb=a-b.$

12. $C=\frac{E}{R}.$

13. $d=\frac{W}{abc}.$

14. $ax=bw.$

15. $V=abc.$

16. $\frac{V_1}{V_2}=\frac{P_2}{P_1}.$

17. $G=\frac{AW}{L}.$

18. $abc=4Rs.$

19. $a=\frac{v_1-v_0}{t}.$

20. $T=\frac{a}{2}(b+b').$

21. $d=\frac{a}{2}(2t+1).$

22. $\frac{mp-rs}{w}=\frac{s-rp}{n}.$

23. $\frac{1}{p}+\frac{1}{p'}=-\frac{2}{r}.$

24. $\frac{\mu_2}{p_2}-\frac{\mu_1}{p_1}=\frac{\mu_2}{f}.$

25. $C=\frac{ne}{r+nR}.$

26. $\mu_1\left(\frac{r-p_1}{p_1}\right)=\mu_2\left(\frac{r-p_2}{p_2}\right).$

27. $W(\theta-t)=wx(T-\theta).$

28. $x\left(\frac{V+l}{v+l}\right)=H\frac{v}{V}.$

EXERCISE XXII

Solve the following systems of equations:

1. $\begin{cases} x-y=3, \\ 2x-y=0. \end{cases}$

2. $\begin{cases} -2x+y=-3, \\ -3x+4y=8. \end{cases}$

3. $\begin{cases} -2x+y=3, \\ 3x+y=13. \end{cases}$

4. $\begin{cases} 2x+3y=46, \\ x+y=18. \end{cases}$

5. $\begin{cases} 4x+y=23, \\ 3x-2y=9. \end{cases}$

6. $\begin{cases} 3x+5y=94, \\ 2x-y=15. \end{cases}$

7. $\begin{cases} 4x+3y=81, \\ -x+2y=21. \end{cases}$

8. $\begin{cases} 4x+2y=38, \\ 3x-3y=6. \end{cases}$

9. $\begin{cases} 2x+y=20, \\ 4x+3y=70. \end{cases}$

10. $\begin{cases} x-9y=0, \\ 4x-y=70. \end{cases}$

11. $\begin{cases} 4x-5y=3, \\ 8x+2y=66. \end{cases}$

12. $\begin{cases} 2x-y=10, \\ 3y+17x=177. \end{cases}$

13. $\begin{cases} 3x-2y=91, \\ 7x+3y=82. \end{cases}$

14. $\begin{cases} 3x-5y=2, \\ 4x+7y=-93. \end{cases}$

15. $\begin{cases} 4x+3y=4, \\ -7x+5y=75. \end{cases}$

16. $\begin{cases} 8x-5y=6, \\ 7x+10y=149. \end{cases}$

17. $\begin{cases} 3x+12y=57, \\ 2x+y=10. \end{cases}$

18. $\begin{cases} 7x+4y=95, \\ x-2y=-7. \end{cases}$

19. $\begin{cases} 27x+14y=41, \\ 36x+51y=87. \end{cases}$

20. $\begin{cases} 100x-143y=757, \\ 11x-91y=8. \end{cases}$

21. $\begin{cases} 55x+31y=171, \\ 27x-11y=18.4. \end{cases}$

22. $\begin{cases} 109x+110y=86, \\ 107x+146y=98. \end{cases}$

23. $\begin{cases} 33x+25y=4, \\ 21x+85y=6. \end{cases}$

24. $\begin{cases} 39x-98y=3, \\ 51x+182y=63. \end{cases}$

25. $$\begin{cases} \dfrac{7}{2x-y} = \dfrac{1}{x-y}, \\ 3x+y=23. \end{cases}$$

26. $$\begin{cases} 9-\frac{1}{2}(x-\frac{1}{2}y)=14, \\ 5+\frac{5}{4}(x+y)=9. \end{cases}$$

27. $$\begin{cases} 5x-(3y-\frac{1}{2})=\frac{3}{4}, \\ 4+x-2(y-\frac{1}{3})=\frac{16}{25}. \end{cases}$$

28. $$\begin{cases} \dfrac{2}{10+x}+\dfrac{1}{10+y}=0, \\ \dfrac{17}{2x-5y}=\dfrac{3}{x+5}. \end{cases}$$

29. $$\begin{cases} \dfrac{x+y}{x-y}=6, \\ \dfrac{x+7}{x+7y}=\dfrac{1}{3}. \end{cases}$$

30. $$\begin{cases} \dfrac{6x-y}{5(x+2y)}=\dfrac{1}{29}, \\ \dfrac{5x+y}{y+1}=2. \end{cases}$$

31. $$\begin{cases} \dfrac{x+y+4}{x-y+6}=6, \\ \dfrac{2x-y+7}{x-2y+7}=-2. \end{cases}$$

32. $$\begin{cases} \dfrac{8x-3y}{5x-2y+3}=4, \\ \dfrac{4x+2y+11}{6x-7y+6}=-\dfrac{4}{3}. \end{cases}$$

33. $$\begin{cases} \dfrac{5x-4}{3}-x=\dfrac{4y+1}{11}+3, \\ \dfrac{3y-7x}{4}+13=\dfrac{x-y}{8}-\dfrac{y}{3}. \end{cases}$$

34. $$\begin{cases} \dfrac{x+1}{4}+\dfrac{y+2}{10}=\dfrac{2(y-x)}{5}, \\ \dfrac{x-1}{4}-\dfrac{y-2}{12}=\dfrac{3y-8x}{18}. \end{cases}$$

35. $$\begin{cases} \dfrac{4x+y-4}{3}+\dfrac{6x+2y-7}{9}=0, \\ \dfrac{2x-y+1}{8}-\dfrac{10x-4y}{3}=1. \end{cases}$$

36.
$$\begin{cases} \dfrac{3x+y+5}{9}+\dfrac{2x-y+5}{6}=\dfrac{x+2y+1}{2}, \\[2ex] \dfrac{2x-y+7}{8}-\dfrac{4x-3y-1}{4}=\dfrac{5x+8y+3}{16}. \end{cases}$$

37.
$$\begin{cases} (3x+8)(4y-3)=(2x+9)(6y-5), \\ (2x-1)(12y-1)=(3x+8)(8y-7). \end{cases}$$

38.
$$\begin{cases} \dfrac{7y}{10}-\dfrac{5y+22}{7}=\dfrac{x}{5}+\dfrac{55-8y}{6}, \\[2ex] \dfrac{3+x}{7}+\dfrac{5x-3y}{11}=\dfrac{1}{11}x+2y-19. \end{cases}$$

39.
$$\begin{cases} \dfrac{2x+7y+5}{7}+3y=29x-\dfrac{4x+11y+5}{8}, \\[2ex] 3x-\dfrac{5x+3y}{11}=\dfrac{11x-14y+241}{154}. \end{cases}$$

40.
$$\begin{cases} 3x+\dfrac{4y+5x}{6}+\dfrac{9x+8y-12}{30}=\dfrac{1}{12}+\dfrac{11x+6y+1}{6}, \\[2ex] \dfrac{x+3y}{7}+\dfrac{7}{12}=\dfrac{4y+5}{8}. \end{cases}$$

41.
$$\begin{cases} x-\dfrac{2y-x}{22}=20-\dfrac{49-2x}{9}, \\[2ex] x+\dfrac{x+7}{9}=25-\dfrac{73-3y}{5}. \end{cases}$$

42.
$$\begin{cases} \dfrac{x-4}{3}+\dfrac{x-3}{4}=\dfrac{8+y}{6}, \\[2ex] \dfrac{2y+7}{8}-\dfrac{3x-y}{7}=\dfrac{3y-2x+4}{8} \end{cases}$$

43.
$$\begin{cases} \dfrac{3y-2}{4}-\dfrac{x-5}{2}=5-\dfrac{2x+3y-1}{8}, \\[2ex] \dfrac{5x+6y-3}{11}-\dfrac{2x+9y-2}{7}=-2. \end{cases}$$

EXERCISE XXIII

Solve the following systems of equations and verify the results :

1. $\begin{cases} x + ay = a^2, \\ x - by = b^2. \end{cases}$

2. $\begin{cases} ax + y = b, \\ x + by = b^2. \end{cases}$

3. $\begin{cases} ax + by = c, \\ \dfrac{x}{y} = \dfrac{m}{n}. \end{cases}$

4. $\begin{cases} a(x+y) + b(x-y) = c, \\ \dfrac{x}{y} = \dfrac{m}{n}. \end{cases}$

5. $\begin{cases} ax + by = c, \\ a_1x + b_1 = c_1y. \end{cases}$

6. $\begin{cases} ax + by = 2, \\ ab(x + y) = a + b. \end{cases}$

7. $\begin{cases} ax = b(y - 2), \\ y - x = \dfrac{a^2 + b^2}{ab}. \end{cases}$

8. $\begin{cases} ay = c(x + 1) - a, \\ y = x + \dfrac{(a+b+c)(c-a)}{ac}. \end{cases}$

9. $\begin{cases} a^2x - b^2y = a + b, \\ bx - ay = -1. \end{cases}$

10. $\begin{cases} ax - (a-b)y = (a-b)^2, \\ bx - y = b(a - b - 1). \end{cases}$

11. $\begin{cases} a(x+c) + b(y-c) = a^2 - b^2, \\ y - x = 2\,c. \end{cases}$

12. $\begin{cases} x - (a+b)y = \dfrac{b^2 - a^2}{b}, \\ (b-a)x + aby = b^2. \end{cases}$

13. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} = \dfrac{2\,a}{a^2 - b^2}, \\ (a+b)x + (a-b)y = a + b. \end{cases}$

14. $\begin{cases} \dfrac{ax}{b} + \dfrac{by}{a} = a + b, \\ \dfrac{x}{a} + \dfrac{y}{b} = \dfrac{a^4 + b^4}{a^2b^2}. \end{cases}$

15. $\begin{cases} \dfrac{x + y}{a + b} = ab, \\ \dfrac{bx + a^2}{ay} = 1 + \dfrac{1}{b^2}. \end{cases}$

16. $\begin{cases} \dfrac{x - a}{y - b} = \dfrac{b + c}{a - c}, \\ \dfrac{x - b}{y - a} = \dfrac{a + c}{b - c}. \end{cases}$

17. $\begin{cases} \dfrac{x}{b + c} - \dfrac{y}{a + c} = a - c, \\ \dfrac{x}{a + b} - \dfrac{y}{b + c} = b - a. \end{cases}$

EXERCISE XXIV

Solve the following systems of equations:

1. $$\begin{cases} 4x + 5y + 9z = 13, \\ 5x + y + 2z = -5, \\ 7x - 5y - 8z = -31. \end{cases}$$

2. $$\begin{cases} x + 5y - 2z = 5, \\ 3x + 8y + 4z = 31, \\ 7x + 25y - 4z = 45. \end{cases}$$

3. $$\begin{cases} 2x - 9y + 10z = 55, \\ 11x - 3y - 5z = 7, \\ 13x - 4y - 6z = 1. \end{cases}$$

4. $$\begin{cases} 5x + 3y - 2z = \frac{9}{8}, \\ 4x - 5y + \dfrac{2z}{3} = -1, \\ 8x - \frac{1}{2}y - z = \frac{7}{12}. \end{cases}$$

5. $$\begin{cases} \frac{1}{2}x + \frac{1}{3}y + \frac{1}{6}z = 14, \\ \frac{1}{3}x + \frac{1}{5}y - \frac{1}{9}z = 5, \\ \frac{1}{4}x + y - \frac{1}{3}z = 12. \end{cases}$$

6. $$\begin{cases} \frac{1}{5}x + 3y - \frac{5}{8}z = 16, \\ 2x - \frac{1}{2}y + \frac{5}{6}z = 25, \\ \frac{1}{3}x - \frac{4}{5}y + z = 17\frac{2}{3}. \end{cases}$$

7. $$\begin{cases} x + y = 2a, \\ ay + z = a^2, \\ bx - z = b^2. \end{cases}$$

8. $$\begin{cases} ax + y = 1, \\ bx + z = b, \\ cz + x = bc. \end{cases}$$

9. $$\begin{cases} bx + ay = 2ab, \\ cy + bz = 2bc, \\ cx + az = 2ac. \end{cases}$$

10. $$\begin{cases} x + ay = a(a+b), \\ a^2z - bx = a^3, \\ y = z - a. \end{cases}$$

11. $$\begin{cases} (b+c)x + by = c, \\ (a+c)y + cz = a, \\ (a+b)z + ax = b. \end{cases}$$

12. $$\begin{cases} ax + y + z = abc + a(b+c), \\ x + by + z = abc + b(a+c), \\ x + y + cz = abc + c(a+b). \end{cases}$$

13. $$\begin{cases} x + y + z + u = 55, \\ x + 2y - z - u = 1, \\ 2x + 3y + 2z - u = 68, \\ 3x - 2y + 2z + u = 54. \end{cases}$$

14. $$\begin{cases} x + 2y + z - u = 10, \\ x - y + 2z + u = 23, \\ x + 3y + 4z - 2u = 39, \\ x - 5y - 4z - 3u = 41. \end{cases}$$

15. $$\begin{cases} x + y + z + u = 10, \\ x + 3y + 5z + 7u = 30, \\ x + 6y + 15z + 28u = 80, \\ x + 10y + 35z + 84u = 188. \end{cases}$$

16. $$\begin{cases} \frac{1}{2}x + \frac{1}{3}y + \frac{1}{5}z - \frac{1}{6}u = 65, \\ \frac{1}{3}x - \frac{1}{4}y + \frac{1}{8}z - \frac{1}{2}u = 8, \\ \frac{1}{4}x - \frac{1}{6}y + \frac{1}{3}z + \frac{1}{5}u = 53, \\ \frac{2}{5}x - \frac{3}{8}y + \frac{3}{4}z - \frac{2}{3}u = 76. \end{cases}$$

17. $$\begin{cases} x + 5y = 23, \\ y + 4z = -1, \\ z + 3u = 20, \\ u + 2v = 3, \\ v + x = 6. \end{cases}$$

18. $$\begin{cases} 2x + 3y = 57, \\ 5x - 4z = 20, \\ 3z + 2u = 48, \\ 4y + 3v = 68, \\ 7u - 6v = 15. \end{cases}$$

19. $$\begin{cases} \dfrac{x}{a+b} + \dfrac{y}{b+c} = b - a, \\ \dfrac{y}{c-a} + \dfrac{z}{c+a} = c + a, \\ \dfrac{x}{b-c} - \dfrac{z}{a-b} = b - c. \end{cases}$$

CHAPTER II

INVOLUTION AND EVOLUTION

INVOLUTION

1. The operation of raising an expression to any given power is called **involution**. An expression is said to be **expanded** when the indicated multiplications have been performed.

Thus, $(a)^2$ and $(a+b)^2$ have been expanded when the respective products have been found to be a^2 and $a^2 + 2ab + b^2$.

MONOMIALS

2. Involution of monomials is subject to the following Index Laws, in the proofs of which $a \neq 0$, and m and n are restricted to positive integers.

I. $(a^m)^n = a^{mn}$.

By definition, $(a^m)^n = [(a \text{ to } m \text{ factors}) \text{ to } n \text{ factors}]$,

by associative law, $= a \text{ to } mn \text{ factors}$,

by definition, $= a^{mn}$.

The exponent of the power of any given monomial is found by multiplying the exponent of the given monomial by the index of the required power.

II. $(ab)^m = a^m b^m$.

By commutative and associative laws,

$(ab)^m = (a \text{ to } m \text{ factors})(b \text{ to } m \text{ factors})$

by definition, $= a^m b^m$.

Similarly, $(abc)^m = a^m b^m c^m$.

The mth power of the product of two quantities is equal to the product of their mth powers.

III. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$.

By commutative and associative laws,

$$\left(\frac{a}{b}\right)^m = (a \text{ to } m \text{ factors}) \div (b \text{ to } m \text{ factors}),$$

by definition, $$= a^m \div b^m = \frac{a^m}{b^m}.$$

The mth power of the quotient of two quantities is the quotient of their mth powers.

3. Involution is also subject to the **Law of Signs.**

$$(-a)(-a) = (-a)^2 = a^2,$$
$$(a)(a) = (a)^2 = a^2,$$
$$(-a)(-a)(-a) = (-a)^3 = -a^3, \text{ etc.}$$

All even powers of a negative monomial are positive, while all odd powers of a negative monomial are negative; all powers of a positive monomial are positive.

EXERCISE XXV

Expand the following expressions:

1. $(a^5)^3$.

2. $(a^4)^3$.

3. $(-a^4)^3$.

4. $(2a)^3$.

5. $-(5m^2)^3$.

6. $-(-4c^2d)^5$.

7. $(2x^2yz^5)^3$.

8. $(-2x^2yz^7)^7$.

9. $-(-4x^5z)^6$.

10. $\left(\dfrac{3x^2y}{7z^2}\right)^2$.

11. $\dfrac{(-11ab^3)^4}{(6a^3b)^5}$.

12. $-\left(\dfrac{-5x^7z}{2ac}\right)^5$.

13. $\left(\dfrac{-10a^5b}{-3xy^4}\right)^7$.

BINOMIALS

4. The expansion of binomials may be shortened by employment of the Binomial Theorem, a proof of which is given in Chapter XII. The use of this theorem is evident from the following type forms, which are derived by multiplication:

$$(a \pm b)^2 = a^2 \pm 2ab + b^2, \tag{1}$$

$$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3, \tag{2}$$

$$(a \pm b)^4 = a^4 \pm 4a^3b + 6a^2b^2 \pm 4ab^3 + b^4, \tag{3}$$

$$(a \pm b)^5 = a^5 \pm 5a^4b + 10a^3b^2 \pm 10a^2b^3 + 5ab^4 \pm b^5, \tag{4}$$

$$(a \pm b)^6 = a^6 \pm 6a^5b + 15a^4b^2 \pm 20a^3b^3 + 15a^2b^4 \pm 6ab^5 + b^6. \tag{5}$$

1. Expand $(a^2 - 2b)^4$.

By type form (3),

$$\begin{aligned}(a^2-2b)^4 &= (a^2)^4 - 4(a^2)^3(2b) + 6(a^2)^2(2b)^2 - 4(a^2)(2b)^3 + (2b)^4\\ &= a^8 - 8a^6b + 24a^4b^2 - 32a^2b^3 + 16b^4.\end{aligned}$$

In a similar way, a polynomial, in the form of a binomial, may be expanded.

EXERCISE XXVI

Expand the following expressions:

1. $(p+q)^4$.
2. $(x+y)^4$.
3. $(1+a)^4$.
4. $(p+q)^5$.
5. $(x-y)^5$.
6. $(b+1)^5$.
7. $(x+y)^6$.
8. $(2a+1)^5$.
9. $(x+2y)^4$.
10. $(x^2-y^2)^4$.
11. $(1-q)^6$.
12. $(2x-3y)^5$.
13. $(3x-2y^2)^4$.
14. $(3mn-4p)^4$.
15. $(2x^2-5y)^5$.
16. $(2a^2-3b^2)^6$.
17. $(a-b+c)^3$.
18. $(a-b-2c)^3$.

EVOLUTION

5. The operation of extracting a root of an expression is called **evolution**, and is indicated by the **radical sign**, $\sqrt{\ }$. The quantities whose roots are to be extracted, called **radicands**, are written after the radical sign. The particular root to be extracted is indicated by a small number, called the **index of the root**, written above the radical sign. The index 2 is generally omitted. If the index of the root is an even number, the root is called an **even root**; if an odd number, the root is called an **odd root**.

Thus, $\sqrt{4}$, $\sqrt[4]{81}$, are even roots; $\sqrt[3]{8}$, $\sqrt[5]{-243}$, are odd roots.

6. If a quantity can be expressed as the product of two equal factors, one of these factors is called the **square root** of the quantity; one of the three equal factors of a quantity is called the **cube root**; and, in general, one of the n equal factors is called the **nth root**.

7. *The one positive root of a positive number is called its* **principal root**; *when n is odd, the one negative root of a negative number is called its* **principal odd root.**

The radical sign will be used throughout this book to indicate the principal roots only, and the letters a, b, c, etc., when used after the radical sign, will always denote positive numbers.

Thus, $\sqrt{4}$ means the positive square root of 4; that is, $\sqrt{4}=+2$; similarly, $\sqrt{25}=+5$; $\sqrt[3]{-27}=-3$; $\sqrt[n]{a^n}=a$.

Note. Only expressions whose exponents are multiples of the indices of the roots will be discussed in this chapter.

MONOMIALS

8. The Index Laws for the evolution of monomials are the inverse forms of the Index Laws for involution.

I. $\sqrt[n]{a^{mn}} = a^m.$

By I, § 2, $(a^m)^n = a^{mn},$

by definition, $\sqrt[n]{a^{mn}} = a^m.$

II. $\sqrt[n]{a^n b^n c^n} = abc.$

By II, § 2, $(abc)^n = a^n b^n c^n,$

by definition, $\sqrt[n]{a^n b^n c^n} = abc.$

From I and II is derived the **Rule for the Root of a Monomial in the form of a Product:** *Divide the exponent of each factor by the index of the required root.*

III. $\sqrt[n]{\dfrac{a^n}{b^n}} = \dfrac{a}{b}.$

By III, § 2, $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n},$

by definition, $\sqrt[n]{\dfrac{a^n}{b^n}} = \dfrac{a}{b}.$

From III is derived the **Rule for the Root of a Monomial in the form of a Quotient:** *Divide the exponent of each factor in the terms of the fraction by the index of the required root.*

1. Simplify $\sqrt[3]{\dfrac{64\,x^3y^6}{343\,a^{12}b^9}}.$

$$\sqrt[3]{\frac{64\,x^3y^6}{343\,a^{12}b^9}} = \sqrt[3]{\frac{2^6x^3y^6}{7^3a^{12}b^9}} = \frac{2^2\,xy^2}{7\,a^4b^3} = \frac{4\,xy^2}{7\,a^4b^3}.$$

EXERCISE XXVII

Simplify the following expressions:

1. $\sqrt{a^2b^2}$.

2. $\sqrt{4 \cdot 9}$.

3. $\sqrt{m^4n^4}$.

4. $\sqrt[3]{a^3b^3c^3}$.

5. $\sqrt[3]{8 \cdot 27}$.

6. $\sqrt[4]{a^8b^4x^{12}}$.

7. $\sqrt[5]{p^5q^5}$.

8. $\sqrt[6]{x^{12}y^{18}}$.

9. $\sqrt{9\,a^2b^4c^6}$.

10. $\sqrt{64\,a^8b^2c^6}$.

11. $\sqrt[3]{27\,a^6b^3}$.

12. $\sqrt[4]{16\,a^4b^8}$.

13. $\sqrt{(a+b)^2}$.

14. $\sqrt{a^2-2\,ab+b^2}$.

15. $\sqrt[3]{(a+b)^3}$.

16. $\sqrt{a^2(x+y)^2}$.

17. $\sqrt[3]{a^3(x-y)^3}$.

18. $\sqrt[3]{-8\,x^6y^{12}}$.

19. $\sqrt{\dfrac{9\,a^4b^6}{16\,x^8}}$.

20. $\sqrt[3]{-\dfrac{64\,a^9b^6}{343\,x^3y^{12}}}$.

21. $\sqrt[5]{-\dfrac{32\,x^{10}y^{25}}{243\,m^{10}n^{15}}}$.

SQUARE ROOT OF POLYNOMIALS

9. Since a polynomial may be squared as a binomial, $(t+u)^2$ may be taken as the type form of the square of a polynomial. Examination of the way that the square root of $t^2+u(2\,t+u)$, which is called **the square root formula**, is obtained, will disclose a method by which the square root of any polynomial may be obtained.

Since $(t+u)^2 = t^2 + 2\,tu + u^2$, $\sqrt{t^2+2\,tu+u^2} = t+u$.

(A) *The first term of the root is the square root of the first term of the formula.*

(B) *The second term of the root is obtained by dividing the second term of the formula by twice the part of the root already found.*

The formula may be applied to any polynomial, if t represents the part of the root already found, and if u represents the next term of the root.

1. Extract the square root of $4x^2 + 4xy + y^2$.

Let $$t^2 + 2tu + u^2 = 4x^2 + 4xy + y^2, \tag{1}$$

by (A), $$t = 2x, \tag{2}$$

squaring (2), $$t^2 = 4x^2, \tag{3}$$

subtracting (3) from (1), $$u(2t + u) = 4xy + y^2, \tag{4}$$

by (B), $$u = y. \tag{5}$$

Substituting $t = 2x$, and $u = y$, in (4),

$$u(2t + u) = y(4x + y) = 4xy + y^2. \tag{6}$$

Since

$$\sqrt{4x^2 + 4xy + y^2} = \sqrt{t^2 + 2tu + u^2} = t + u, \tag{7}$$

and since $t = 2x$, and $u = y$, $$\sqrt{4x^2 + 4xy + y^2} = 2x + y. \tag{8}$$

The work may be more compactly written:

$$\begin{array}{c|l}
 & 4x^2 + 4xy + y^2 \underline{|\, 2x + y} \\
t = 2x & 4x^2 \\
\hline
2t = 4x & \quad 4xy + y^2 \\
u = y & \\
2t + u = 4x + y & \\
\underline{u(2t + u) = y(4x + y)} & \quad \underline{4xy + y^2}
\end{array}$$

The terms of the polynomial should be arranged either in ascending or in descending order of some one of its letters; otherwise the formula method is not available.

If the polynomial contains more than three terms, it should be carefully noticed that the part of the root already found in every case is represented by t.

Since $(a + b - c)^2 = [a + (b - c)]^2 = [(a + b) - c]^2$,

and since $(t + u)^2 = (a + b - c)^2$,

t is represented successively by a and $a + b$.

2. Extract the square root of $a^2+4c^2+b^2-2ab+4bc-4ac$.

$$
\begin{array}{r|l}
 & \qquad\qquad\qquad\qquad\qquad \underline{|\,a-b-2c} \\
 & a^2-2ab-4ac+b^2+4bc+4c^2 \\
t=a & a^2 \\
\hline
2t=2a & \quad -2ab-4ac+b^2+4bc+4c^2 \\
u=-b & \\
2t+u=2a-b & \\
u(2t+u)=-b(2a-b) & \quad -2ab \qquad\quad +b^2 \\
\hline
2t=2a-2b & \qquad\qquad -4ac \qquad +4bc+4c^2 \\
u=-2c & \\
2t+u=2a-2b-2c & \\
u(2t+u)=-2c(2a-2b-2c) & \qquad\qquad -4ac \qquad +4bc+4c^2 \\
\hline
\end{array}
$$

In the above example, after the second term of the root has been found, the first two terms are together equal to t. Since $t=(a-b)$, and t has been squared and subtracted, the remainder again corresponds to the expression $u(2t+u)$.

EXERCISE XXVIII

Extract the square roots of the following expressions:

1. $25a^2-70ac+49c^2$.
2. $a^2+2ab+b^2+2ac+2bc+c^2$.
3. $b^2+2bc+c^2-2ab-2ac+a^2$.
4. $4a^2+12ab+9b^2+16ac+24bc+16c^2$.
5. $49a^2+4b^2-28ab+42ac+9c^2-12bc$.
6. $9a^4+30a^3b+49a^2b^2+40ab^3+16b^4$.
7. $89a^2b^2-70ab^3+16a^4-56a^3b+25b^4$.
8. $\dfrac{a^2}{b^2}-\dfrac{2a}{c}+\dfrac{b^2}{c^2}+\dfrac{2c}{b}-\dfrac{2b}{a}+\dfrac{c^2}{a^2}$.
9. $\dfrac{9a^4}{b^6}+\dfrac{8b^2}{3}+\dfrac{4b^6}{9a^2}-\dfrac{6a^3}{b^4}+\dfrac{13a^2}{b^2}+\dfrac{8b^4}{3a}$.

CHAPTER III

RADICALS

10. The quantity $\sqrt[n]{a}$ has already been defined, § 6, as the quantity whose nth power is a, or $(\sqrt[n]{a})^n = a$. If a is an exact nth power, the existence of such a quantity is at once evident, as $\sqrt[3]{8} = 2$. But if a is not an exact nth power, it becomes necessary to prove the existence of $\sqrt[n]{a}$. Such a proof is beyond the province of this book; and a simple numerical example must suffice. It is not possible to obtain exactly the value of $\sqrt{2}$, since there is no number, integral or fractional, whose square is exactly 2. But,

$$(1.4)^2 < 2 < (1.5)^2, \tag{A}$$

$$(1.41)^2 < 2 < (1.42)^2, \tag{B}$$

$$(1.414)^2 < 2 < (1.415)^2. \tag{C}$$

In (A), since 2 lies between $(1.4)^2$ and $(1.5)^2$, $\sqrt{2}$ differs from 1.4 and 1.5 by less than they differ from each other; that is, since 1.4 and 1.5 differ from each other by 0.1, $\sqrt{2}$ differs from either by less than 0.1; similarly in (B), $\sqrt{2}$ differs from 1.41 and 1.42 by less than 0.01; and in (C), $\sqrt{2}$ differs from 1.414 and 1.415 by less than 0.001. Continuing the process shown in (A), (B), and (C), a number may be found which will represent as close an approximation of $\sqrt{2}$ as is required.

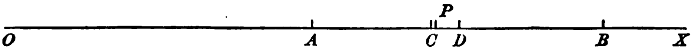


FIG. 1

The value of $\sqrt{2}$ may be represented graphically. On the line *OX*, Fig. 1, let equal distances be laid off from *O* toward the right, and *OA* represent the number 1, *OB* the number 2, etc. Then 1.4 will be represented by *OC*, 1.5 by *OD*. The numbers 1.4, 1.41, 1.414 will be seen to be represented by lines whose terminal points move toward the right, while the numbers 1.5, 1.42, 1.415 will be represented by lines whose terminal points move toward the left. The terminal points representing these two sets of numbers will approach each other, but no terminal point in either set can cross into the region of the other. Yet the numbers show that the terminal points may be made as near to each other as may be required. There will be some point *P* which will be the limiting position of both sets of terminal points; and the line *OP* will represent $\sqrt{2}$.

11. An indicated root of a quantity is called a **radical.**

Thus, $\sqrt{a}$, $\sqrt[3]{27}$, are radicals.

An expression which is composed of radicals is called **a radical expression.**

Thus, $\sqrt{x}+\sqrt[3]{27}$, $\sqrt{a}-\sqrt{b}$, are radical expressions.

All integers and fractions are called **rational quantities.**

All other numbers are called **irrational quantities.** The simplest class of irrational quantities consists of indicated roots which cannot be extracted.

Thus, 2, and $\frac{5}{8}$, are rational; $\sqrt{2}$, $\sqrt{1+\sqrt{2}}$, are irrational.

An expression which contains rational quantities only is called a **rational expression.**

Thus, $a+\frac{2}{3}$ is a rational expression.

An expression which contains an irrational quantity is called an **irrational expression.**

Thus, $a+\sqrt{2}$ is an irrational expression.

12. A radical whose radicand is rational and whose root is irrational is called a **surd.**

Thus, $\sqrt{a}$ and $\sqrt[3]{4}$ are surds; while $\sqrt{1+\sqrt{3}}$, being the indicated root of a quantity not rational, is not a surd.

The **order of a surd** depends upon the index of the root. A **quadratic surd,** or a surd of the second order, has 2 for the index of the root; a **cubic surd,** or a surd of the third order, has 3 for the index of the root; a **biquadratic surd,** or a surd of the fourth order, has 4 for the index of the root, etc.

Thus, $\sqrt{a}$, $\sqrt[3]{b}$, $\sqrt[4]{c}$, are respectively quadratic, cubic, and biquadratic surds.

13. A rational factor of a surd is called the **coefficient** of the surd.

Thus, $\frac{3}{4}$ is the coefficient of $\frac{3}{4}\sqrt{ax}$.

Surds which have 1 as a coefficient, expressed or implied, are called **entire surds.**

Thus, $\sqrt{ay}$ and $\sqrt[3]{4}$ are entire surds.

Surds which have other coefficients than 1 are called **mixed surds.**

Thus, $2\sqrt{x}$ and $3\sqrt{a-b}$ are mixed surds.

A surd is called a **monomial surd** if it consists of a single surd.

Thus, $\sqrt[3]{x^2}$ and $5\sqrt{3}$ are monomial surds.

The sum of a rational, and a surd quantity, or the sum of two monomial surds, is called a **binomial surd.**

14. The difference between algebraic and arithmetical irrational quantities should be noticed. Such a quantity as $\sqrt{2}$ is an arithmetical irrational quantity; similarly, quantities such as $\sqrt{a}$ are considered algebraic irrational quantities, although if $a = 4$, $\sqrt{a}$ is an arithmetical rational quantity.

In this, as in the preceding chapter, the **principal roots** only are discussed, and the quantity under the radical sign is restricted to positive values.

Thus, $\sqrt{4} \neq \pm 2$, but $\sqrt{4} = 2$. This fact must be kept in mind, for otherwise some of the proofs of the principles will not hold.

PRINCIPLES OF RADICALS

15. I. *The product of the nth roots of any number of quantities is equal to the nth root of their products.*

By II, § 2, $(\sqrt[n]{a}\sqrt[n]{b}\sqrt[n]{c})^n = abc,$

by definition, $\sqrt[n]{a}\sqrt[n]{b}\sqrt[n]{c} = \sqrt[n]{abc}.$

If the radicand contains a factor whose exponent is a multiple of the index of the root, the surd may be simplified by I. Since $\sqrt[n]{a^n} = a$,

by I, $\sqrt[n]{a^n b} = \sqrt[n]{a^n}\sqrt[n]{b} = a\sqrt[n]{b}.$

1. Simplify $\sqrt[3]{16}$.

$$\sqrt[3]{16} = \sqrt[3]{2^4} = \sqrt[3]{2^3}\sqrt[3]{2} = 2\sqrt[3]{2}.$$

2. Simplify $\sqrt[4]{32x^5y}$.

$$\sqrt[4]{32x^5y} = \sqrt[4]{2^5x^5y} = \sqrt[4]{2^4x^4}\sqrt[4]{2xy} = 2x\sqrt[4]{2xy}.$$

3. Simplify $\sqrt{25a - 25b}$.

$$\sqrt{25a - 25b} = \sqrt{25(a - b)} = \sqrt{5^2}\sqrt{a - b} = 5\sqrt{a - b}.$$

EXERCISE XXIX

Simplify the following surds:

1. $\sqrt{48}$.
2. $\sqrt{150}$.
3. $\sqrt[3]{16}$.
4. $\sqrt[3]{54}$.
5. $\sqrt[4]{80}$.
6. $\sqrt[5]{96}$.
7. $\sqrt[6]{192}$
8. $\sqrt{25\,ax^2}$.
9. $\sqrt{5\,m^2n^2}$.
10. $\sqrt[3]{27\,xy^8}$.
11. $\sqrt[5]{y^9}$.
12. $\sqrt{16\,x-16\,y}$.
13. $\sqrt[4]{x^7(a-b)}$.
14. $2\sqrt[3]{27\,a-54\,b}$.
15. $3\sqrt[3]{(a-b)^4}$.

16. The coefficient of a radical may be introduced under the radical by raising the coefficient to the power indicated by the index of the root. Since $a=\sqrt[n]{a^n}$,

$$a\sqrt[n]{b}=\sqrt[n]{a^n}\sqrt[n]{b}=\sqrt[n]{a^nb}.$$

Reduce the mixed surds $2\,a\sqrt{2\,b}$, and $-3\,ax\sqrt[3]{a^2x}$, to entire surds.

$$2\,a\sqrt{2\,b}=\sqrt{(2\,a)^2}\sqrt{2\,b}=\sqrt{8\,a^2b}.$$

$$-3\,ax\sqrt[3]{a^2x}=\sqrt[3]{(-3\,ax)^3}\sqrt[3]{a^2x}=\sqrt[3]{-27\,a^5x^4}.$$

EXERCISE XXX

Reduce the following mixed surds to entire surds:

1. $2\sqrt[4]{5}$.
2. $3\sqrt[5]{2}$.
3. $-5\sqrt[3]{4}$.
4. $-2\,a\sqrt[3]{x}$.
5. $-ab\sqrt[5]{bx}$.
6. $a-b\sqrt{a-b}$.
7. $(a-b)\sqrt{a-b}$.
8. $-(a-x)\sqrt[3]{x-y}$.

17. When the radicand is in the fractional form, it may be made integral, by 1, § 15. Since $\sqrt[n]{\left(\frac{1}{a}\right)^n}=\frac{1}{a}$,

$$\sqrt[n]{\frac{b}{a^n}}=\sqrt[n]{\frac{1}{a^n}}\sqrt[n]{b}=\frac{1}{a}\sqrt[n]{b}.$$

When the denominator of a radicand in the fractional form does not contain a quantity whose exponent is a multiple of the index of the root, the denominator may be put into such a form that the indicated root may be extracted by multiplying both numerator and denominator by the same quantity.

Thus,

$$\sqrt{\frac{a}{b}}=\sqrt{\frac{a}{b}\left(\frac{b}{b}\right)}=\sqrt{\frac{ab}{b^2}}=\sqrt{\frac{1}{b^2}}\sqrt{ab}=\frac{1}{b}\sqrt{ab}.$$

1. Simplify $\sqrt{\frac{2}{3}}$.

$$\sqrt{\frac{2}{3}}=\sqrt{\frac{2}{3}\left(\frac{3}{3}\right)}=\sqrt{\frac{2\cdot 3}{3^2}}=\sqrt{\frac{1}{3^2}}\sqrt{6}=\frac{1}{3}\sqrt{6}.$$

2. Simplify $\sqrt[3]{\frac{2\,xy}{9\,ab}}$.

$$\sqrt[3]{\frac{2\,xy}{9\,ab}}=\sqrt[3]{\frac{2\,xy}{3^2\,ab}}=\sqrt[3]{\frac{2\,xy}{3^2\,ab}\left(\frac{3\,a^2b^2}{3\,a^2b^2}\right)}$$

$$=\sqrt[3]{\frac{1}{3^3a^3b^3}}\sqrt[3]{6\,a^2b^2xy}=\frac{1}{3\,ab}\sqrt[3]{6\,a^2b^2xy}.$$

EXERCISE XXXI

Simplify the following surds :

1. $\sqrt{\frac{1}{2}}$.
2. $\sqrt[3]{\frac{1}{2}}$.
3. $\sqrt[4]{\frac{1}{6}}$.
4. $\sqrt{\frac{a}{x}}$.
5. $\sqrt[3]{\frac{a}{x}}$.
6. $\sqrt[3]{\frac{3a}{4b}}$.
7. $\sqrt{\frac{2a}{3b}}$.
8. $\sqrt[3]{\frac{5a^2}{16}}$.
9. $\sqrt[3]{\frac{2a}{125c^2}}$.
10. $\sqrt[4]{\frac{5a}{7b^2}}$.
11. $\sqrt[3]{\frac{3a}{7c^4}}$.
12. $\sqrt{\frac{x^2y}{12a^3b}}$.
13. $\sqrt[3]{\frac{5a}{16x^4y}}$.
14. $\sqrt[4]{\frac{7m}{162x^3y^3}}$.
15. $\sqrt[5]{\frac{3a^3}{2048x^7y}}$.
16. $\sqrt[3]{\frac{21abc}{250x^3y}}$.
17. $\sqrt{\frac{a^5c}{242x^7z}}$.
18. $\sqrt[3]{\frac{32a^5b^7}{243mx}}$.
19. $\sqrt[4]{\frac{10a^6c^2}{81x^7y^2}}$.
20. $\sqrt[3]{\frac{15x^7y^8}{16(a-b)^4}}$.

18. II. *The quotient of the nth roots of two quantities is equal to the nth root of their quotient, or* $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$.

By III, § 2, $\left(\frac{\sqrt[n]{a}}{\sqrt[n]{b}}\right)^n = \frac{a}{b}$,

by definition, $\frac{\sqrt[n]{a}}{\sqrt{b}} = \sqrt[n]{\frac{a}{b}}$.

Express $\sqrt{2a} \div \sqrt{3b}$ as a single radical.

$$\sqrt{2a} \div \sqrt{3b} = \sqrt{\frac{2a}{3b}} = \sqrt{\frac{2a}{3b}\left(\frac{3b}{3b}\right)}$$

$$= \sqrt{\frac{1}{3^2b^2}}\sqrt{2 \cdot 3ab} = \frac{1}{3b}\sqrt{6ab}.$$

EXERCISE XXXII

In the following radicals, simplify the quotients expressed as single radicals:

1. $\sqrt{6} \div \sqrt{5}$.
2. $\sqrt[3]{4} \div \sqrt[3]{3}$.
3. $\sqrt[4]{a} \div \sqrt[4]{b}$.
4. $\sqrt[5]{x} \div \sqrt[5]{y}$.
5. $\sqrt{3\,ab} \div \sqrt{7\,xy}$.
6. $\sqrt[5]{x^2y} \div \sqrt[5]{16\,m^4n}$.
7. $\sqrt[3]{49\,x^2y^2} \div \sqrt[3]{13\,bc}$.
8. $\sqrt[3]{(a-b)^2} \div \sqrt[3]{2\,a}$.
9. $\sqrt[4]{(m-n)^8} \div \sqrt[4]{8\,mn}$.
10. $\sqrt{32\,xy} \div \sqrt{27\,ab^5}$.

19. III. *The nth power of the mth root of any quantity is equal to the mth root of the nth power, or* $(\sqrt[m]{a})^n = \sqrt[m]{a^n}$.

By definition, $(\sqrt[m]{a})^n = \sqrt[m]{a}$ to n factors,

by I, § 15, $= \sqrt[m]{a^n}$.

Simplify $(\sqrt[3]{2\,a^5c})^4$.

$$(\sqrt[3]{2\,a^5c})^4 = \sqrt[3]{(2\,a^5c)^4} = \sqrt[3]{2^4a^{20}c^4} = \sqrt[3]{2^3a^{18}c^3}\,\sqrt[3]{2\,a^2c} = 2\,a^6c\,\sqrt[3]{2\,a^2c}.$$

EXERCISE XXXIII

Simplify the following radicals:

1. $(\sqrt{2})^3$.
2. $(\sqrt[3]{3})^2$.
3. $(\sqrt[4]{2})^3$.
4. $(\sqrt{2\,a})^3$.
5. $(\sqrt[4]{3\,a})^3$.
6. $(\sqrt[5]{2\,a^2bx})^2$.
7. $(\sqrt[7]{7})^3$.
8. $(\sqrt[3]{a+b})^2$.
9. $(\sqrt[3]{4\,ab^2})^4$.
10. $(\sqrt[3]{54\,a^4})^2$.
11. $(\sqrt{8\,xy^5})^3$.
12. $(\sqrt[6]{9\,x^2y^4})^3$.

20. IV. *The mth root of the nth root of any quantity is equal to the mnth root of the quantity, or* $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$.

Let $$x = \sqrt[m]{\sqrt[n]{a}}, \tag{1}$$

raising both members of (1) to mth power,

$$x^m = \sqrt[n]{a}, \tag{2}$$

raising both members of (2) to nth power,

$$x^{mn} = a, \tag{3}$$

extracting mnth roots in (3),

$$x = \sqrt[mn]{a}, \tag{4}$$

from (4), $$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}. \tag{5}$$

A surd is said to be in its **simplest form** when neither of the reductions explained under I, §§ 15, 17, and IV, § 20, may be applied; that is, when the radicand is integral and contains no factor whose exponent is a multiple of the index of the root, and when the index of the root is as small as possible. **Similar surds** are those which, when reduced to their simplest form, differ in their coefficients only. Surds which are not similar are called **dissimilar surds**.

Thus, $\frac{1}{3}\sqrt{a}$ and $b\sqrt{a}$ are similar surds, and $\frac{1}{3}\sqrt{a}$ and $b\sqrt[3]{a}$ are dissimilar surds.

Reduce $\sqrt[6]{27}$ and $\sqrt[15]{243\,a^5}$ to their simplest forms.

$$\sqrt[6]{27} = \sqrt[6]{3^3} = \sqrt{\sqrt[3]{3^3}} = \sqrt{3}.$$

$$\sqrt[15]{243\,a^5} = \sqrt[15]{3^5\,a^5} = \sqrt[3]{\sqrt[5]{3^5\,a^5}} = \sqrt[3]{3\,a}.$$

EXERCISE XXXIV

Reduce the following radicals to their simplest forms:

1. $\sqrt[6]{4}$.
2. $\sqrt[6]{25}$.
3. $\sqrt[12]{16}$.
4. $\sqrt[15]{64}$.
5. $\sqrt[10]{100}$.
6. $\sqrt[10]{243}$.
7. $\sqrt[6]{a^2}$.
8. $\sqrt[12]{a^3}$.
9. $\sqrt[3]{\sqrt{a}}$.
10. $\sqrt[5]{\sqrt{a}}$.
11. $\sqrt[5]{\sqrt[3]{a^2}}$.
12. $\sqrt[3]{\sqrt[4]{ab}}$.

21. V. *The nth root of any quantity is equal to the mnth root of the mth power of the quantity, or* $\sqrt[n]{a} = \sqrt[mn]{a^m}$.

Since $\sqrt[n]{a} = \sqrt[m]{(\sqrt[n]{a})^m}$,

by III, § 19, $= \sqrt[m]{\sqrt[n]{a^m}}$,

by IV, § 20, $= \sqrt[mn]{a^m}$.

By V, surds of different orders may be reduced to equivalent surds of the same order — that order being the least common multiple of the original orders.

Reduce $\sqrt{2}$ and $\sqrt[3]{5}$ to equivalent surds of the same order.

$$\sqrt{2} = \sqrt[6]{(2)^3} = \sqrt[6]{8}.$$
$$\sqrt[3]{5} = \sqrt[6]{(5)^2} = \sqrt[6]{25}.$$

Since $\sqrt[6]{25} > \sqrt[6]{8}$, $\sqrt[3]{5} > \sqrt{2}$.

EXERCISE XXXV

Reduce the following surds to equivalent surds of the same order:

1. $\sqrt{a}$, $\sqrt[4]{a^3}$.
2. $\sqrt{a}$, $\sqrt[3]{a}$.
3. $\sqrt{2x}$, $\sqrt[3]{4x^2}$.
4. $\sqrt[3]{a^2b}$, $\sqrt[4]{ab^2}$.
5. $\sqrt{3a^2}$, $\sqrt[5]{2xy}$.
6. $\sqrt[3]{7x^2y}$, $\sqrt[4]{3ab^2}$.
7. $4\sqrt{2x^2y}$, $\sqrt[6]{2xy^2}$.
8. $\sqrt[5]{8x^3}$, $\sqrt[4]{27y^3}$.
9. $\sqrt[6]{5a^4b^3}$, $\sqrt[5]{3a^4b^3}$.
10. $\sqrt{11axy}$, $\sqrt[3]{4a^2xy^2}$.

Arrange the following surds in order of magnitude:

11. $\sqrt[3]{4}$, $\sqrt[4]{8}$.

12. $\sqrt{3}$, $\sqrt[3]{2}$, $\sqrt[4]{5}$.

13. $\sqrt[6]{9}$, $\sqrt[12]{10}$, $\sqrt[24]{100}$.

14. $\sqrt{12}$, $\sqrt[3]{48}$, $\sqrt[6]{81}$.

15. $\sqrt{\frac{1}{2}}$, $\sqrt[3]{\frac{1}{4}}$, $\sqrt[4]{\frac{1}{8}}$.

16. $\sqrt{\frac{3}{4}}$, $\sqrt[3]{\frac{4}{5}}$, $\sqrt[4]{\frac{5}{6}}$.

ADDITION AND SUBTRACTION OF RADICALS

22. Similar radicals may be added or subtracted by writing the algebraic sum of their coefficients as the coefficient of the common radical, taken as the unit of addition. Each radical should be reduced to its simplest form before the process of addition or subtraction is begun. Addition or subtraction of dissimilar radicals may only be indicated by connecting them with the proper signs.

1. Simplify $\sqrt{a^3} - 2a\sqrt{a^5} + 3\sqrt{a}$.

$$\begin{array}{rr} \sqrt{a^3} = \sqrt{a^2}\,\sqrt{a} = & a\sqrt{a}, \\ -2a\sqrt{a^5} = -2a\sqrt{a^4}\,\sqrt{a} = & -2a^3\sqrt{a}, \\ 3\sqrt{a} = & 3\sqrt{a}, \\ \hline \sqrt{a^3} - 2a\sqrt{5} + 3\sqrt{a} = & (a - 2a^3 + 3)\sqrt{a}. \end{array}$$

2. Simplify $2\sqrt{\frac{1}{2}} - 4\sqrt{\frac{1}{8}} + 8\sqrt{2}$.

$$\begin{array}{rr} 2\sqrt{\dfrac{1}{2}} = 2\sqrt{\dfrac{1}{2^2}}\sqrt{2} = & \sqrt{2}, \\ -4\sqrt{\dfrac{1}{8}} = -4\sqrt{\dfrac{1}{2^4}}\,\sqrt{2} = & -\sqrt{2}, \\ 8\sqrt{2} = & 8\sqrt{2}, \\ \hline 2\sqrt{\frac{1}{2}} - 4\sqrt{\frac{1}{8}} + 8\sqrt{2} = & 8\sqrt{2}. \end{array}$$

EXERCISE XXXVI

Simplify the following radical expressions:

1. $4a\sqrt[5]{a^2b} - \sqrt[5]{32a^7b} - a\sqrt[5]{243a^2b}$.

2. $a\sqrt{x} - \sqrt{a^3x} + 7a\sqrt{a^4x}$.

3. $\sqrt[3]{27x^2y} + \sqrt[3]{8x^2y} - \sqrt[3]{125x^2y}$.

4. $\sqrt{16a} + \sqrt{81a} + \sqrt{144a^2b^2}$.

5. $\sqrt{a^2bc} + \sqrt{ab^2c} + \sqrt{abc^2}$.

6. $\sqrt{a^3bc} + 2\sqrt{ab^3c} - 5\sqrt{abc^3}$.

7. $\sqrt{9a^5b^3c} + \sqrt{25a^3b^5c} + \sqrt{36a^3b^3c^3}$.

8. $\sqrt{9x} - \sqrt{16x} + 3\sqrt{25x} - \sqrt{100x}$.

9. $\sqrt{18x} + \sqrt{147x} - 2\sqrt{32x} - \sqrt{192x} + \sqrt{72x}$.

10. $3\sqrt{x(a+b^2)} - \sqrt{4a^2x} - \sqrt{25b^2x} - \sqrt{(a+b)^2x}$.

11. $b\sqrt{a^5b} - (a+b)\sqrt{a^3b^3} + a\sqrt{ab^5} + b\sqrt{a^7b^7}$.

12. $\sqrt[3]{a^3b} + \sqrt[3]{bc^3} + \sqrt[3]{b(a-c)^3} + \sqrt[3]{a^3b^4c^3}$.

13. $\sqrt[3]{54} - \sqrt[3]{128} + \sqrt[3]{250}$.

14. $\sqrt{\frac{1}{2}} + \frac{3}{2}\sqrt{2} - \sqrt{\frac{2}{9}}$.

15. $\frac{2}{3}\sqrt{3} - \sqrt{\frac{3}{25}} + \frac{8}{5}\sqrt{\frac{1}{3}}$.

16. $3\sqrt[3]{\frac{1}{9}} - 2\sqrt[3]{81} + 6\sqrt[3]{3}$.

17. $2a\sqrt[4]{a} - \sqrt[4]{\dfrac{1}{a^3}} - 8\sqrt[4]{a^5}$.

18. $\sqrt{\frac{1}{32}} - \frac{5}{2}\sqrt{8} + 4\sqrt{\frac{1}{8}}$.

19. $\sqrt{\dfrac{x-a}{x+a}} + \sqrt{x^2-a^2} + \sqrt{(x^2-a^2)^3}$.

MULTIPLICATION OF RADICALS

23. The product of any two monomial radicals may be found by applying V, § 21, and I, § 15.

1. Multiply $2\sqrt{5}$ by $3\sqrt[3]{4}$.

$2\sqrt[2]{5^4} \cdot 3\sqrt[3]{4}$

$$2\sqrt{5} = 2\sqrt[6]{5^3} = 2\sqrt[6]{125},$$
$$3\sqrt[3]{4} = 3\sqrt[6]{2^4} = 3\sqrt[6]{16}.$$
$$2\sqrt{5} \cdot 3\sqrt[3]{4} = 6\sqrt[6]{2000}.$$

2. Multiply $3\sqrt[3]{ax^2}$ by $4\sqrt[4]{bx^3}$.

$$3\sqrt[3]{ax^2} = 3\sqrt[12]{(ax^2)^4} = 3\sqrt[12]{a^4x^8},$$
$$4\sqrt[4]{bx^3} = 4\sqrt[12]{(bx^3)^3} = 4\sqrt[12]{b^3x^9},$$
$$3\sqrt[3]{ax^2} \cdot 4\sqrt[4]{bx^3} = 12\sqrt[12]{a^4b^3x^{17}} = 12\sqrt[12]{x^{12}}\,\sqrt[12]{a^4b^3x^5} = 12x\sqrt[12]{a^4b^3x^5}.$$

EXERCISE XXXVII

Find the products of the following radicals:

1. $\sqrt{a} \cdot \sqrt[4]{b}$.
2. $\sqrt{x} \cdot \sqrt[6]{y}$.
3. $\sqrt[3]{b} \cdot \sqrt[12]{c}$.
4. $\sqrt[3]{5} \cdot \sqrt[6]{2}$.
5. $\sqrt[3]{a} \cdot \sqrt[4]{b}$.
6. $\sqrt{a} \cdot \sqrt[3]{x}$.
7. $\sqrt[4]{a} \cdot \sqrt[6]{b}$.
8. $\sqrt[6]{x} \cdot \sqrt[12]{y}$.
9. $\sqrt{x^6} \cdot \sqrt[20]{x}$.
10. $\sqrt[4]{a^7} \cdot \sqrt[10]{a^3}$.
11. $\sqrt{2} \cdot \sqrt[3]{3}$.
12. $\sqrt[3]{12} \cdot \sqrt[4]{2}$.
13. $\sqrt[4]{x} \cdot \sqrt[6]{x^3}$.
14. $\sqrt[5]{x^7} \cdot \sqrt[10]{y}$.
15. $\sqrt{2a^9} \cdot \sqrt[3]{3x}$.
16. $\sqrt[4]{a^7x^5} \cdot \sqrt[12]{2x}$.
17. $\sqrt[4]{\frac{4}{5}} \cdot \sqrt[6]{\frac{5}{2}}$.
18. $\sqrt[4]{\frac{25}{7}} \cdot \sqrt[3]{\frac{14}{5}}$.
19. $\sqrt{\frac{a}{b}} \cdot \sqrt[4]{\frac{b^3}{a}}$.
20. $\sqrt[3]{\frac{a^5}{b^2}} \cdot \sqrt[9]{\frac{b^3}{a^4}}$.
21. $\sqrt[4]{72} \cdot \sqrt[6]{108}$.
22. $\sqrt[4]{5a} \cdot \sqrt[5]{2a^6}$.
23. $\sqrt[3]{5b^2x} \cdot \sqrt[4]{10x}$.
24. $\sqrt[3]{1024} \cdot \sqrt[4]{\frac{1}{16}}$.
25. $\sqrt[4]{1024a^3} \cdot \sqrt[5]{2}$.

MULTIPLICATION OF POLYNOMIALS INVOLVING RADICALS

24. The product of two or more polynomials involving radicals is found in the same way that the product of two rational expressions is found; the terms of the polynomials being expressed in their simplest form before the process of multiplication is attempted.

Multiply $\sqrt{60}+\sqrt{24}$ by $\sqrt{27}-\sqrt{8}$.

$$\begin{aligned} \sqrt{60}+\sqrt{24} &= 2\sqrt{15}+2\sqrt{6}, \\ \sqrt{27}-\sqrt{8} &= 3\sqrt{3}-2\sqrt{2}, \end{aligned}$$

$$\begin{array}{l} 2\sqrt{15}+2\sqrt{6} \\ \underline{\;3\sqrt{3}-2\sqrt{2}\qquad\qquad\qquad\qquad} \\ 6\sqrt{45}+6\sqrt{18}-4\sqrt{30}-4\sqrt{12}= \\ 18\sqrt{5}+18\sqrt{2}-4\sqrt{30}-8\sqrt{3}. \end{array}$$

EXERCISE XXXVIII

Multiply the following expressions involving radicals:

1. $2a\sqrt{x}-6\sqrt{a^3xy}$ by $2\sqrt{a^3x}+5a\sqrt{y}$.
2. $4\sqrt{8}-\sqrt{32}+2\sqrt{50}$ by $\sqrt{2}$.
3. $\sqrt{75}-\sqrt{150}+2\sqrt{243}$ by $\sqrt{3}$.
4. $\sqrt{\frac{1}{5}}+\sqrt{\frac{1}{6}}+\sqrt{\frac{1}{8}}$ by $\sqrt{5}+\sqrt{6}+\sqrt{8}$.
5. $2\sqrt{27}-3\sqrt{75}+4\sqrt{12}$ by $2\sqrt{3}+5\sqrt{48}+4\sqrt{5}$.
6. $3\sqrt{12}+\sqrt{32}+2\sqrt{80}$ by $4\sqrt{45}-2\sqrt{48}-2\sqrt{18}$.
7. $3\sqrt{27}-5\sqrt{20}-2\sqrt{343}$ by $2\sqrt{48}-3\sqrt{45}-3\sqrt{112}$.
8. $3\sqrt{18}+4\sqrt{54}-4\sqrt{75}$ by $4\sqrt{48}+2\sqrt{162}-3\sqrt{150}$.
9. $2\sqrt[3]{16}+4\sqrt[3]{24}-3\sqrt[3]{108}$ by $2\sqrt[3]{54}-3\sqrt[3]{81}+4\sqrt[3]{32}$.
10. $4\sqrt[3]{24}+3\sqrt[3]{256}-5\sqrt[3]{135}$ by $3\sqrt[3]{81}-\sqrt[3]{32}+2\sqrt[3]{625}$.

DIVISION OF RADICALS

25. The quotient of any two monomial radicals may be found by applying V, § 21, and II, § 18.

Divide $\sqrt[3]{12}$ by $\sqrt{2}$; and $3\sqrt{2x}$ by $2\sqrt[4]{x^3}$.

$$\frac{\sqrt[3]{12}}{\sqrt{2}} = \frac{\sqrt[6]{(2^2 \cdot 3)^2}}{\sqrt[6]{2^3}} = \sqrt[6]{\frac{2^4 \cdot 3^2}{2^3}} = \sqrt[6]{2 \cdot 3^2} = \sqrt[6]{18}.$$

$$\frac{3\sqrt{2x}}{2\sqrt[4]{x^3}} = \frac{3\sqrt[4]{(2x)^2}}{2\sqrt[4]{x^3}} = \frac{3}{2}\sqrt[4]{\frac{2^2x^2}{x^3}} = \frac{3}{2}\sqrt[4]{\frac{2^2x^2}{x^3}\left(\frac{x}{x}\right)} = \frac{3}{2}\sqrt[4]{\frac{1}{x^4}}\sqrt[4]{4x^3} = \frac{3}{2x}\sqrt[4]{4x^3}.$$

EXERCISE XXXIX

Find the quotient of the following radicals:

1. $\sqrt{a} \div \sqrt[4]{b}$.

2. $\sqrt[3]{x^2} \div \sqrt[6]{x}$.

3. $\sqrt[3]{a} \div \sqrt[4]{b}$.

4. $\sqrt{a} \div \sqrt[3]{a}$.

5. $\sqrt[4]{a} \div \sqrt[6]{a}$.

6. $\sqrt[4]{a^3} \div \sqrt[5]{a^4}$.

7. $\sqrt[3]{x^2} \div \sqrt[6]{x^5}$.

8. $\sqrt{30} \div \sqrt[3]{45}$.

9. $a\sqrt{a} \div \sqrt[3]{a^2}$.

10. $\sqrt[6]{\frac{a^5}{bc}} \div \sqrt[9]{\frac{a^7}{b^2c}}$.

11. $3\sqrt{5} \div \sqrt[4]{15}$.

12. $12\sqrt{27} \div \sqrt[3]{9}$.

13. $4\sqrt[3]{49} \div 3\sqrt{7}$.

14. $4\sqrt[3]{100} \div \sqrt{10}$.

15. $3\sqrt[6]{90} \div 2\sqrt[3]{18}$.

16. $3\sqrt{343} \div \sqrt[3]{49}$.

17. $\sqrt[16]{\frac{x^3y^9}{z^7}} \div \sqrt[12]{\frac{x^5y^{11}}{z^{10}}}$.

18. $2\sqrt[3]{686} \div 5\sqrt{56}$.

19. $10\sqrt{216} \div 2\sqrt[3]{36}$.

20. $\sqrt[4]{1024} \div 2\sqrt[3]{192}$.

21. $\sqrt[3]{\frac{(a-b)^2}{(x-y)^2}} \div \sqrt[4]{\frac{(a-b)^3}{(x-y)^7}}$.

22. $\sqrt[4]{\frac{x^5}{(a-b)^6}} \div \sqrt[5]{\frac{x^3}{(a-b)^8}}$.

DIVISION OF POLYNOMIALS INVOLVING RADICALS

26. The quotient of a polynomial involving radicals by a monomial radical may be found in the same way as in rational expressions.

The division of two polynomials involving radicals is best effected by a process called division by means of **rationalization**; by this process the radical denominator is transformed into a rational quantity. The simplest factor, the product of which and the given radical produces a rational quantity, is called the **rationalizing factor.**

Thus, the rationalizing factors of $\sqrt{2}$, $\sqrt[3]{ax}$, $\sqrt[4]{a^3}$ are respectively $\sqrt{2}$, $\sqrt[3]{a^2x^2}$, $\sqrt[4]{a}$.

Two binomial quadratic surds in the forms $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$ are called **conjugates** of each other. Since $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y}) = x - y$, the rationalizing factor of a quadratic binomial surd is its conjugate. It is necessary to multiply both the terms of the fraction expressing the quotient, by the rationalizing factor.

1. Rationalize the denominator of $\frac{2}{\sqrt{3}}$.

$$\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2}{3}\sqrt{3}.$$

2. Rationalize the denominator of $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$.

$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} \cdot \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}} = \frac{a+2\sqrt{ab}+b}{a-b}.$$

EXERCISE XL

Rationalize the denominators of the following fractions:

1. $\frac{5}{\sqrt{5}}$.

2. $\frac{12}{5\sqrt{3}}$.

3. $\frac{48}{5\sqrt{32}}$.

4. $\frac{a+b}{\sqrt{a+b}}$.

5. $\frac{a^2-b^2}{\sqrt{a-b}}$.

6. $\frac{a}{\sqrt{\frac{a}{b}}}$.

7. $\frac{3\sqrt{7}}{7\sqrt{3}}$.

8. $\frac{6\sqrt{5}}{5\sqrt{3}}$.

9. $\frac{a}{b\sqrt{ab}}$.

10. $\frac{4\sqrt{18}}{5\sqrt{12}}$.

11. $\frac{1}{3+\sqrt{2}}$.

12. $\frac{1}{a+\sqrt{b}}$.

13. $\frac{a\sqrt{b}}{\sqrt{b}+c}$.

14. $\frac{a}{\sqrt{b}+\sqrt{c}}$.

15. $\frac{a+\sqrt{b}}{a-\sqrt{b}}$.

16. $\frac{10+\sqrt{21}}{\sqrt{7}+\sqrt{3}}$.

17. $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$.

18. $\frac{\sqrt{8}-\sqrt{6}}{\sqrt{8}+\sqrt{6}}$.

19. $\frac{a^2-a\sqrt{b}+b}{a-\sqrt{b}}$.

20. $\frac{a-b}{\sqrt{a+2b}-\sqrt{3b}}$.

21. $\frac{xy}{\sqrt{1-x}+\sqrt{1-y}}$.

SQUARE ROOT OF A BINOMIAL QUADRATIC SURD

27. The square root of a binomial quadratic surd, in the form of a perfect square, may often be extracted by inspection if it be remembered that the binomial quadratic surd is a disguised form of $a^2 + 2ab + b^2$. Since $(\sqrt{a} + \sqrt{b})^2 = a + 2\sqrt{ab} + b = (a + b) + 2\sqrt{ab}$, the square root of $(a + b) + 2\sqrt{ab}$ is found by obtaining two quantities whose sum is $a + b$ and whose product is ab. Since $(\sqrt{2} + \sqrt{3})^2 = 2 + 2\sqrt{6} + 3 = 5 + 2\sqrt{6}$, the square root of $5 + 2\sqrt{6}$ is $\sqrt{2} + \sqrt{3}$.

The binomial quadratic surd may be still further disguised by the introduction of the coefficient of $\sqrt{ab}$ under the radical, thus: $(a + b) + \sqrt{4ab}$. Before finding the square root it is necessary that the term corresponding to $2\sqrt{ab}$ shall be written with the coefficient 2.

1. Extract the square root of $8 + \sqrt{60}$.

$$\sqrt{8 + \sqrt{60}} = \sqrt{8 + 2\sqrt{15}} = \sqrt{3} + \sqrt{5}.$$

2. Extract the square root of $\sqrt{24} + \sqrt{25}$.

$$\sqrt{\sqrt{24} + \sqrt{25}} = \sqrt{2\sqrt{6} + 5} = \sqrt{3} + \sqrt{2}.$$

EXERCISE XLI

Extract the square root of the following binomial quadratic surds:

1. $3 + 2\sqrt{2}$.
2. $4 + 2\sqrt{3}$.
3. $7 + 2\sqrt{10}$.
4. $8 - 2\sqrt{15}$.
5. $6 + 2\sqrt{5}$.
6. $9 - 2\sqrt{14}$.
7. $11 - 2\sqrt{24}$.
8. $11 - 2\sqrt{28}$.
9. $\sqrt{121} - 2\sqrt{10}$.
10. $\sqrt{81} - \sqrt{80}$.
11. $\sqrt{121} - \sqrt{120}$.
12. $\sqrt{64} - \sqrt{28}$.
13. $\sqrt{256} - \sqrt{156}$.
14. $\frac{15}{4} - \frac{1}{2}\sqrt{14}$.
15. $2 - \sqrt{3}$.

RADICAL EQUATIONS

28. An equation which involves the indicated root of the unknown is called a **radical** or **irrational equation.**

Thus, $\sqrt{3+x}=2$ is a radical equation.

A radical equation which involves square roots only can often be solved as a simple equation by isolating one or more of the radicals and rationalizing the resulting equation by squaring. But since two equations with different signs may give the same result when squared, the solution obtained by solving the squared equation does not necessarily satisfy the given equation. *It is necessary to test the solution in every case by substituting in the given equation.*

If the equation contains a single radical, it is simpler to isolate the radical and then square the resulting equation; if the equation contains two or more radicals, the more involved radical is isolated. The squared equation should then be simplified, especial care being taken to reduce the resulting equivalent equation to the simplest integral form.

1. Solve the equation: $\sqrt{x+6}+\sqrt{x-2}=4.$ (1)

Transposing in (1), $\sqrt{x+6}=4-\sqrt{x-2},$ (2)

squaring (2), $x+6=16-8\sqrt{x-2}+x-2,$ (3)

transposing and uniting in (3),

$$8\sqrt{x-2}=8, \quad (4)$$

dividing (4) by 8, $\sqrt{x-2}=1,$ (5)

squaring (5), $x-2=1,$ (6)

transposing and uniting in (6), $x=3.$ (7)

VERIFICATION: $\sqrt{9}+\sqrt{1}=4$; therefore 3 is a root of (1).

2. Solve the equation: $\sqrt{x+6}-\sqrt{x-2}=4.$ (1)

Transposing in (1), $\sqrt{x+6}=4+\sqrt{x-2},$ (2)

squaring (2), $x+6=16+8\sqrt{x-2}+x-2,$ (3)

transposing and uniting in (3),

$8\sqrt{x-2}=-8,$ (4)

dividing (4) by 8, $\sqrt{x-2}=-1,$ (5)

squaring (5), $x-2=1,$ (6)

transposing and uniting in (6), $x=3.$ (7)

Substituting in (1), $\sqrt{9}-\sqrt{1}\neq 4$; therefore 3 is *not* a root of (1).

3. Solve the equation: $(a-b)\sqrt{\dfrac{x}{a-b}}+b=a.$ (1)

Transposing in (1), $(a-b)\sqrt{\dfrac{x}{a-b}}=a-b,$ (2)

dividing (2) by $a-b$, $\sqrt{\dfrac{x}{a-b}}=1,$ (3)

squaring (3), $\dfrac{x}{a-b}=1,$ (4)

multiplying (4) by $a-b$, $x=a-b.$ (5)

VERIFICATION: $(a-b)\sqrt{\dfrac{a-b}{a-b}}+b=a;\ a-b+b=a.$

4. Solve the equation: $\sqrt{2+x}+\sqrt{x-3}=\sqrt{4x-3}.$ (1)

Squaring (1), $2+x+2\sqrt{x^2-x-6}+x-3=4x-3,$ (2)

transposing and uniting in (2), $2\sqrt{x^2-x-6}=2x-2,$ (3)

dividing (3) by 2, $\sqrt{x^2-x-6}=x-1,$ (4)

squaring (4), $x^2-x-6=x^2-2x+1,$ (5)

transposing and uniting in (5), $x=7.$ (6)

VERIFICATION: $\sqrt{9}+\sqrt{4}=\sqrt{25}.$

EXERCISE XLII

Solve the following radical equations:

1. $\sqrt{x+5}=3.$

2. $6\sqrt{x+4}=11.$

3. $7=3\sqrt{x}-4.$

4. $\sqrt{7x+2}=4.$

5. $\sqrt{5+x}=5-\sqrt{x}.$

6. $\sqrt{15+x}=3\sqrt{5}-\sqrt{x}.$

7. $\sqrt{2x+11}+\sqrt{2x-5}=8.$

8. $\sqrt{27x+1}=2-3\sqrt{3x}.$

9. $\sqrt{4+x\sqrt{24+x^2}}=x+2.$

10. $\dfrac{\sqrt{x}+\sqrt{a}}{\sqrt{x}-\sqrt{a}}=b.$

11. $\sqrt{x}+\sqrt{a+x}=\dfrac{b}{\sqrt{a+x}}.$

12. $\sqrt{x}+\sqrt{3+x}=\dfrac{24}{\sqrt{3+x}}.$

13. $\sqrt{x+4ab}=2a+\sqrt{x}.$

14. $\sqrt{x+a}=a-\sqrt{x-a}.$

15. $b-a\sqrt{x}=\sqrt{a^2x}.$

16. $\dfrac{\sqrt{x+1}}{\sqrt{x+3}}=\dfrac{\sqrt{x+3}}{\sqrt{x+6}}.$

17. $x=a-\sqrt{a^2-x\sqrt{x^2+8a^2}}.$

18. $\sqrt{5+2x}=\sqrt{2(8+9x)}-\sqrt{1+8x}.$

19. $3\sqrt{1+2x}-\sqrt{8x-15}=\sqrt{2(x+6)}.$

20. $\sqrt{9x-14}+3\sqrt{x+2}=2\sqrt{9x-2}.$

CHAPTER IV

IMAGINARY AND COMPLEX NUMBERS

29. In the previous chapter the quantity after the radical sign has been restricted to positive values. If the index of the root is odd, a negative radicand causes no difficulty; for, if n is odd, $\sqrt[n]{-a} = -\sqrt[n]{a}$, since $(\sqrt[n]{-a})^n = -a$, and $(-\sqrt[n]{a})^n = -a$, if n is odd. For example, $\sqrt[3]{-8} = -2$. But there is difficulty, when n is even, for $\sqrt{-4}$ can not be expressed as a positive or negative number; nor can any approximation to its value be obtained. It is, therefore, not contained in the real system of numbers used in the earlier chapters of this book, and a new set of operations must be defined, and a new set of laws obtained, if we are to make use of such expressions in our operations. For this purpose we make the following definitions and assumptions.

The even roots of negative numbers are called **pure imaginary numbers.**

The number $\sqrt{-1}$ is called the **imaginary unit** and is represented by the letter **i**.

The number $\sqrt{-a^2}$ is defined as equal to $a\sqrt{-1}$, or ai.

The sum of a real number, a, and a pure imaginary number, bi, can only be indicated, $a + bi$. Such numbers are called **complex numbers.**

Two complex numbers are equal only when their real and their imaginary parts are equal, or, if $a + bi = c + di$, then $a = c$ and $b = d$.

30. We shall assume that complex numbers, which have been reduced to their typical form, $a + bi$, may be added, subtracted, multiplied, and divided just as real numbers, the symbol i being used as any letter, subject only to the substitution of the following values for its various powers, after the operations indicated in the problem have been performed:

$$\begin{aligned} i^2 &= -1, \\ i^3 &= i \cdot i^2 = -i, \\ i^4 &= i \cdot i^3 = +1, \\ i^5 &= i \cdot i^4 = +i, \text{ etc.} \end{aligned}$$

According to these assumptions

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

Also $\sqrt{-a}\ \sqrt{-b} = i\sqrt{a} \cdot i\sqrt{b} = i^2\sqrt{ab} = -\sqrt{ab}.$

This last operation should be carefully noted. The product of $\sqrt{-a}$ and $\sqrt{-b}$ is one of the square roots of ab, but it is not the principal square root. All difficulty will be avoided, if in any computation we replace $\sqrt{-a}$ by $i\sqrt{a}$ before performing any of the indicated operations.

1. Reduce $\sqrt{-16}$, $\sqrt{-x^6}$, $\sqrt{-4x^5y}$ to the typical form.

$$\begin{aligned} \sqrt{-16} &= \sqrt{16}\,i = 4i, \\ \sqrt{-x^6} &= \sqrt{x^6}\,i = x^3 i, \\ \sqrt{-4x^5y} &= \sqrt{2^2x^4}\ \sqrt{xy}\,i = 2x^2\sqrt{xy}\,i. \end{aligned}$$

2. Add $\sqrt{-16}$, $\sqrt{-25}$, and $\sqrt{-36}$.

$$\begin{aligned} \sqrt{-16} &= \sqrt{16}\,i = 4i, \\ \sqrt{-25} &= \sqrt{25}\,i = 5i, \\ \sqrt{-36} &= \sqrt{36}\,i = 6i, \end{aligned}$$

$$\sqrt{-16} + \sqrt{-25} + \sqrt{-36} = (4 + 5 + 6)i = 15i.$$

3. Multiply $\sqrt{-4}+\sqrt{2}$ by $2\sqrt{-4}+3\sqrt{2}$.

$$\begin{array}{l}
2\,i+\sqrt{2} \\
\underline{4\,i+3\sqrt{2}\quad} \\
8\,i^2+4\sqrt{2}\,i \\
\underline{\qquad\quad +6\sqrt{2}\,i+6} \\
8\,i^2+10\sqrt{2}\,i+6 \\
=-8+10\sqrt{2}\,i+6=10\sqrt{2}\,i-2.
\end{array}$$

31. Conjugate imaginaries. Two complex numbers, $a+bi$ and $a-bi$, which differ only in the sign between the real and the imaginary part are called **conjugate imaginaries.**

The sum and the product of any two conjugate imaginaries are real numbers;

for $$(a+bi)+(a-bi)=2\,a,$$

and $$(a+bi)(a-bi)=a^2+b^2.$$

A fraction in the form $\frac{a+bi}{c+di}$ can be reduced to the typical form by multiplying both numerator and denominator by the conjugate of the denominator.

Simplify $\frac{2-3\,i}{5+2\,i}$.

$$\frac{2-3\,i}{5+2\,i}=\frac{(2-3\,i)(5-2\,i)}{(5+2\,i)(5-2\,i)}=\frac{4-19\,i}{25+4}=\frac{4}{29}-\frac{19}{29}\,i.$$

EXERCISE XLIII

Reduce the following pure imaginary quantities to the typical forms:

1. $\sqrt{-25}$. **4.** $\sqrt{-a^2}$. **7.** $\sqrt{-225\,y^4}$.

2. $\sqrt{-36}$. **5.** $\sqrt{-x^2y^2}$. **8.** $\sqrt{-484\,x^8}$.

3. $\sqrt{-100}$. **6.** $\sqrt{-4\,x^6y^8}$. **9.** $\sqrt{-625\,x^8y^2}$.

Simplify the following expressions:

10. $\sqrt{-16}+\sqrt{-25}+\sqrt{-64}-5\sqrt{-100}$.

11. $\sqrt{-49}+\sqrt{-121}-\sqrt{-169}-\sqrt{-196}$.

12. $\sqrt{-x^2}-\sqrt{-4x^2}-\sqrt{-9x^2}+\sqrt{-25x^2}$.

13. $3+\sqrt{-4}+5\sqrt{-16}+16+7\sqrt{-225}$.

14. $a+b\sqrt{-x^2}+2a-b\sqrt{-x^2}+3a-4b\sqrt{-x^2}$.

15. $\sqrt{-4}\cdot\sqrt{-9}$.

16. $\sqrt{-25}\cdot\sqrt{-36}$.

17. $\sqrt{-9}\cdot\sqrt{-16}$.

18. $\sqrt{-\frac{1}{4}}\cdot\sqrt{-\frac{1}{9}}$.

19. $-\sqrt{-x^2}\cdot\sqrt{-x^4}$.

20. $\sqrt{-a^3b^2}\cdot\sqrt{-ab^4}$.

21. $(\sqrt{-a}+\sqrt{-b})(\sqrt{-a}-\sqrt{-b})$.

22. $(\sqrt{-x}+\sqrt{-y})(2\sqrt{-x}+3\sqrt{-y})$.

23. $\sqrt{-a^3b}\cdot\sqrt{-a^2b^2}\cdot\sqrt{-ab^3}$.

24. $\sqrt{-2}\cdot\sqrt{-3}\cdot\sqrt{-6}$.

25. $(1+\sqrt{-4})^2$.

26. $(\sqrt{-x}+\sqrt{y})^2$.

27. $(2\sqrt{-9}+3\sqrt{4})^2$.

28. $(3\sqrt{-9}-4\sqrt{-4})^2$.

29. $(1-\sqrt{-4})^3$.

30. $(\sqrt{-9}+1)^3$.

31. $(1+\sqrt{-16})^4$.

32. $(1+\sqrt{-1})^5$.

33. $(\sqrt{-\frac{1}{4}}+\sqrt{\frac{1}{9}})^3$.

34. $(\sqrt{-\frac{1}{16}}-\sqrt{-\frac{1}{9}})^3$.

35. $(\sqrt{-4}+\sqrt{-9}+\sqrt{-16})^2$.

36. $(\sqrt{-x^2}-\sqrt{-y^2}-\sqrt{-z^2})^2$.

37. $\dfrac{\sqrt{-4}}{\sqrt{-9}}$.

38. $\dfrac{1}{\sqrt{-16}}$.

39. $\dfrac{3}{\sqrt{-9}}$.

40. $\dfrac{9}{\sqrt{-9}}$.

41. $\dfrac{5+\sqrt{-3}}{1+\sqrt{-1}}$.

42. $\dfrac{\sqrt{a}}{\sqrt{-a}}$.

43. $\dfrac{\sqrt{x}}{\sqrt{-x^3}}$.

44. $\dfrac{\sqrt{-49}}{-\sqrt{-25}}$.

45. $\dfrac{1}{1+\sqrt{-2}}$.

46. $\dfrac{4-3\sqrt{-5}}{3+2\sqrt{-5}}$.

47. $\dfrac{2}{2+\sqrt{-9}}$.

48. $\dfrac{a+\sqrt{-b}}{a-\sqrt{-b}}$.

49. $\dfrac{a-\sqrt{-b}}{a+\sqrt{-b}}$.

50. $\dfrac{1}{\sqrt{2}+\sqrt{-2}}$.

51. $\dfrac{3-5\sqrt{-3}}{5+3\sqrt{-3}}$.

52. $(2\sqrt{3}-\sqrt{-5})(4\sqrt{3}-2\sqrt{-5})$.

53. $(x-5+2\sqrt{-1})(x-5-2\sqrt{-1})$.

54. $(\sqrt{x}-2+\sqrt{-3})(\sqrt{x}+2-\sqrt{-3})$.

55. $(x-2\sqrt{5}+3\sqrt{-5})(x-2\sqrt{5}-3\sqrt{-5})$.

56. $(2-\sqrt{-3}-3\sqrt{-2})(4\sqrt{-3}+6\sqrt{-2})$.

57. $(x-\frac{1}{2}+\frac{1}{2}\sqrt{-3})(x-\frac{1}{2}-\frac{1}{2}\sqrt{-3})$.

58. $(x-2-\sqrt{3})(x-2+\sqrt{3})(x-3+\sqrt{-1})$
$(x-3-\sqrt{-1})$.

59. $(x-1-\sqrt{-2})(x-1+\sqrt{-2})(x-2+\sqrt{-3})$
$(x-2-\sqrt{-3})$.

CHAPTER V

THEORY OF EXPONENTS

THE EXPONENT IN THE FORM OF A POSITIVE FRACTION

32. In § 8 it was shown, if m and n are integers and n is a multiple of m, that $\sqrt[m]{a^n} = a^{\frac{n}{m}}$. If, however, n is not an exact multiple of m, there can be no meaning attached to $a^{\frac{n}{m}}$ according to the definition of an integral exponent. Thus, it is impossible to speak of $a^{\frac{3}{4}}$ as meaning a taken three-fourths of a time as a factor. The definition of an exponent is therefore extended to include the exponent $\frac{n}{m}$, it being understood that $a^{\frac{n}{m}}$ (where n and m are positive integers and a is a positive real quantity) is simply an alternative way of writing $\sqrt[m]{a^n}$, or the *principal* value of the mth root of the nth power of a.

This extension of the definition of an exponent is convenient only in case exponents in the form of a positive fraction conform to the laws of exponents which have been shown to hold for positive integers. That is, exponents in the form of a positive fraction must be shown to obey the laws,

$$a^x a^y = a^{x+y}, \qquad \text{I}$$

$$a^x \div a^y = a^{x-y}, \qquad \text{II}$$

$$(a^x)^y = a^{xy}, \qquad \text{III}$$

$$(ab)^x = a^x b^x. \qquad \text{IV}$$

33. I. $a^{\frac{n}{m}}a^{\frac{r}{s}} = a^{\frac{n}{m}+\frac{r}{s}}.$

By definition, $a^{\frac{n}{m}}a^{\frac{r}{s}} = \sqrt[m]{a^n}\sqrt[s]{a^r},$

by V, § 21, $= \sqrt[ms]{a^{ns}}\sqrt[ms]{a^{mr}},$

by I, § 15, $= \sqrt[ms]{a^{ns+mr}},$

by definition, $= a^{\frac{ns+mr}{ms}},$

or, $= a^{\frac{n}{m}+\frac{r}{s}}.$

II. $a^{\frac{n}{m}} \div a^{\frac{r}{s}} = a^{\frac{n}{m}-\frac{r}{s}}.$

By definition, $a^{\frac{n}{m}} \div a^{\frac{r}{s}} = \sqrt[m]{a^n} \div \sqrt[s]{a^r},$

by V, § 21, $= \sqrt[ms]{a^{ns}} \div \sqrt[ms]{a^{mr}},$

by II, § 18, $= \sqrt[ms]{a^{ns-mr}},$

by definition, $= a^{\frac{ns-mr}{ms}},$

or, $= a^{\frac{n}{m}-\frac{r}{s}}.$

III. $(a^{\frac{n}{m}})^{\frac{r}{s}} = a^{\frac{nr}{ms}}.$

By definition, $a^{\frac{n}{m}} = \sqrt[m]{a^n},$

by conditions, $(a^{\frac{n}{m}})^{\frac{r}{s}} = (\sqrt[m]{a^n})^{\frac{r}{s}},$

by definition, $= \sqrt[s]{(\sqrt[m]{a^n})^r}.$

by III, § 19, $= \sqrt[s]{\sqrt[m]{a^{nr}}},$

by IV, § 20, $= \sqrt[ms]{a^{nr}},$

by definition, $= a^{\frac{nr}{ms}}.$

IV. $(ab)^{\frac{n}{m}} = a^{\frac{n}{m}}b^{\frac{n}{m}}$.

By definition, $(ab)^{\frac{n}{m}} = \sqrt[m]{(ab)^n}$,

by III, § 19, $= \sqrt[m]{a^nb^n}$,

by I, § 15, $= \sqrt[m]{a^n}\,\sqrt[m]{b^n}$,

by definition, $= a^{\frac{n}{m}}b^{\frac{n}{m}}$.

It may also be shown that $a^{\frac{n}{m}} = a^{\frac{kn}{km}}$.

By definition, $a^{\frac{n}{m}} = \sqrt[m]{a^n}$,

by V, § 21, $= \sqrt[km]{a^{kn}}$,

by definition, $= a^{\frac{kn}{km}}$.

ZERO AS AN EXPONENT

34. It is now found convenient to extend the definition of an exponent so as to include, not only all positive integers and fractions, but also all negative integers and fractions and zero, so that any rational number may be used as an exponent.

In defining negative and zero exponents it is natural to give them such a meaning that they will conform to the index laws.

If the first of these laws, $a^na^m = a^{n+m}$, is to hold for a zero exponent,

$$a^0a^n = a^n, \text{ or } a^0 = 1.$$

Hence we shall define *the zero power of any number as unity.*

The remaining laws also hold when this meaning is attached to a^0, for

$$a^n \div a^0 = a^n \div 1 = a^n = a^{n-0},$$

and $$(a^0)^n = 1^n = 1 = a^{0 \cdot n},$$

and $$(ab)^0 = 1 = a^0b^0.$$

NEGATIVE INTEGERS AS EXPONENTS

35. Again, if the first law is to hold when one of the exponents is negative,

$$a^{-n}a^n = a^{-n+n} = a^0 = 1.$$

Hence $$a^{-n} = \frac{1}{a^n}.$$

We shall, therefore, define *a number with a negative exponent as equal to the reciprocal of the same number with an equal positive exponent.*

36. It is necessary to show that negative integral exponents conform to all the laws of exponents.

I. $$a^{-p}a^{-q} = a^{-p-q}.$$

For $$a^{-p}a^{-q} = \frac{1}{a^p} \cdot \frac{1}{a^q} = \frac{1}{a^{p+q}} = a^{-p-q}.$$

II. $$a^{-p} \div a^{-q} = a^{-p+q}.$$

For $$a^{-p} \div a^{-q} = \frac{1}{a^p} \div \frac{1}{a^q} = \frac{a^q}{a^p} = a^{-p+q}.$$

III. $$(a^{-p})^{-q} = a^{pq}.$$

For $$(a^{-p})^{-q} = \frac{1}{(a^{-p})^q} = \frac{1}{\left(\frac{1}{a^p}\right)^q} = \frac{1}{\frac{1}{a^{pq}}} = a^{pq}.$$

IV. $(ab)^{-p} = a^{-p}b^{-p}.$

$$(ab)^{-p} = \frac{1}{(ab)^p} = \frac{1}{a^pb^p} = \frac{1}{a^p} \cdot \frac{1}{b^p} = a^{-p}b^{-p}.$$

NEGATIVE FRACTIONS AS EXPONENTS

37. If m is a positive integer and n is a negative integer which is not an exact multiple of m, a being a real quantity, $a^{-\frac{n}{m}}$ may be defined as the alternative form of $\sqrt[m]{a^{-n}}$.

Let the student show by combining the results of § 33 and § 36 that exponents in the form of a negative fraction conform to the index laws.

Let him also show that laws I, II, and III hold in case one of the exponents is a positive and the other a negative integer or fraction.

EXAMPLES

1. $a^{\frac{3}{4}} = \sqrt[4]{a^3}$.

2. $a^{\frac{2}{3}} \cdot a^{-\frac{3}{4}} \cdot a^{-1} \cdot a^{\frac{3}{2}} = a^{\frac{8-9-12+18}{12}} = a^{\frac{5}{12}}$.

3. $25^{\frac{1}{2}} \cdot 8^{-\frac{2}{3}} = (5^2)^{\frac{1}{2}} \cdot (2^3)^{-\frac{2}{3}} = 5 \cdot 2^{-2} = 5 \cdot \frac{1}{2^2} = 5 \cdot \frac{1}{4} = \frac{5}{4}$.

4. $\left(\frac{16\,a^0b^{-2}}{c^{-6}}\right)^{\frac{1}{4}} = \left(\frac{2^4 \cdot 1 \cdot b^{-2}}{c^{-6}}\right)^{\frac{1}{4}} = \left(\frac{2^4c^6}{b^2}\right)^{\frac{1}{4}} = \frac{2\,c^{\frac{3}{2}}}{b^{\frac{1}{2}}}$.

5. $\left(\frac{3^{-3}a^{-\frac{3}{2}}b^6}{64}\right)^{-\frac{1}{3}} = \left(\frac{3^{-3}a^{-\frac{3}{2}}b^6}{2^6}\right)^{-\frac{1}{3}} = \frac{3\,a^{\frac{1}{2}}b^{-2}}{2^{-2}} = \frac{3 \cdot 2^2a^{\frac{1}{2}}}{b^2} = \frac{12\,a^{\frac{1}{2}}}{b^2}$.

EXERCISE XLIV

Change each of the following expressions into radicals.

1. $a^{\frac{1}{2}}$.
2. $a^{\frac{3}{4}}$.
3. $x^{0.3}$.
4. $9^{\frac{3}{2}}$.
5. $a^{\frac{1}{n}}$.
6. $a^{-\frac{4}{5}}$.
7. $36^{-\frac{1}{2}}$.
8. $\left(\frac{4}{25}\right)^{-\frac{3}{2}}$.
9. $3\,a^{\frac{1}{5}} + b^{\frac{1}{8}}$.
10. $5\,b^{-\frac{7}{8}} - 2\,c^{\frac{1}{4}}$.
11. $(x+y)^{\frac{1}{4}}$.
12. $3(a+2\,b)^{-\frac{2}{5}}$.

Change each of the following radicals into expressions containing exponents in the form of fractions:

13. $\sqrt[3]{a^2}$.

14. $\sqrt[n]{a}$.

15. $\sqrt[m]{2x^p}$.

16. $7a\sqrt[4]{b^8}$.

17. $\sqrt[3]{a+b}$.

18. $\sqrt{3a^{-1}}$.

19. $\sqrt[3]{4x^{-4}}$.

20. $\sqrt[n]{9a^{-x}}+\sqrt[m]{b^{-n}}$.

21. $\sqrt[5n]{4x^{-10}}$.

22. $\sqrt[4]{a^{p-q}}$.

23. $\sqrt[4]{x-y}$.

24. $\sqrt[3]{(a+b)^{-2}}$.

Free each of the following expressions from negative exponents:

25. a^{-2}.

26. $3a^{-\frac{1}{2}}$.

27. $9c^{-6}$.

28. $2a^{-5}b^2$.

29. $\dfrac{5a^0b^{-3}}{2c^{-5}}$.

30. $\dfrac{16^{-2}x^{-\frac{5}{2}}y^2}{4a^{-\frac{3}{2}}b^{-\frac{2}{3}}}$.

31. $\dfrac{2a^{-1}b^{-3}c^{-4}}{5x^{-2}y^{-4}z^{-6}}$.

32. $\dfrac{a^0c^{-5}d^{-\frac{5}{3}}}{9x^{-\frac{3}{4}}y^{-\frac{1}{m}}}$.

33. $\dfrac{25^{-2}x(a-b)^{-3}}{4a^{-2}(x-y)^{-4}}$.

Find the value of each of the following expressions:

34. $9^{\frac{3}{2}}$.

35. $64^{\frac{5}{6}}$.

36. $(100^2)^{\frac{1}{2}}$.

37. $81^{\frac{3}{4}}$.

38. $3(49)^{\frac{1}{2}}$.

39. $2(81)^{\frac{3}{4}}$.

40. $2\cdot 16^{-\frac{1}{4}}$.

41. $216^{-\frac{2}{3}}$.

42. $243^{-\frac{3}{5}}$.

43. $256^{-\frac{3}{4}}$.

44. $64^{-\frac{2}{3}}$.

45. $(27+5)^{-\frac{4}{5}}$.

Find the product of each of the following expressions:

46. $a^8\cdot a^{-2}$.

47. $x^{-7}\cdot x^{-10}$.

48. $3x^{-11}\cdot 4x^0$.

49. $a\cdot a^{-1}$.

50. $5b^m\cdot 4b^{-m}$.

51. $a^{m-3}\cdot a^{3-2m}$.

52. $(3a)^{-5}\cdot(3a)^5$.

53. $5x\cdot 5x^{-1}$.

54. $(a+b)^{-3}\cdot(a+b)^5$.

55. $(-a)^{-m}\cdot(-a)^{4+m}$.

56. $(-ax)^{-m-n}\cdot(-ax)^{m-n}$.

57. $4^{n+1}\cdot 2^{2-n}$.

Find the quotient of each of the following expressions:

58. $b^{-6} \div b^{-8}$.

59. $x^{-5} \div x$.

60. $a^{-7} \div a^{0}$.

61. $a^{-m+1} \div a^{-m-1}$.

62. $4x \div 3x^{1-n}$.

63. $a^{-2m-5n} \div a^{-m-6n}$.

64. $a^{-4m}b^{4-m} \div a^{m-1}b^{4}$.

65. $-a \div 3a^{-4}$.

66. $7a^{-2x}b^{3y} \div c^{m}d^{-n}$.

67. $8a^{-x}b^{-y^2} \div a^{3x}b^{5y^2}$.

38. The index laws which apply to monomials apply to the terms of a polynomial.

1. Multiply $a^{\frac{2}{3}} - a^{\frac{1}{3}} + 1 - a^{-\frac{1}{3}}$ by $a^{\frac{1}{3}} + 1 - a^{-\frac{1}{3}}$.

$$\begin{array}{llllll}
a^{\frac{2}{3}} - a^{\frac{1}{3}} + 1 - a^{-\frac{1}{3}} & & & & & \\
\underline{a^{\frac{1}{3}} + 1 \;\; - a^{-\frac{1}{3}}} & & & & & \\
a & - a^{\frac{2}{3}} & + a^{\frac{1}{3}} & - 1 & & \\
 & a^{\frac{2}{3}} & - a^{\frac{1}{3}} & + 1 & - a^{-\frac{1}{3}} & \\
 & & - a^{\frac{1}{3}} & + 1 & - a^{-\frac{1}{3}} & + a^{-\frac{2}{3}} \\
\hline
a & & - a^{\frac{1}{3}} & + 1 & - 2a^{-\frac{1}{3}} & + a^{-\frac{2}{3}} = a - a^{\frac{1}{3}} + 1 - \dfrac{2}{a^{\frac{1}{3}}} + \dfrac{1}{a^{\frac{2}{3}}}.
\end{array}$$

In this chapter, unless the contrary is stated, results are to be written with exponents in the form of positive integers or positive fractions.

2. Divide $x^{2}\sqrt[3]{y^{-4}} - 2 + x^{-1}\sqrt[3]{y^{2}}$ by $\sqrt{x}\sqrt[3]{y^{-1}} - \sqrt{x^{-1}}\sqrt[3]{y}$.

$$\begin{array}{r|l}
x^{2}y^{-\frac{4}{3}} - 2 + x^{-1}y^{\frac{2}{3}} & x^{\frac{1}{2}}y^{-\frac{1}{3}} - x^{-\frac{1}{2}}y^{\frac{1}{3}} \\
\cline{2-2}
\underline{x^{2}y^{-\frac{4}{3}} - xy^{-\frac{2}{3}}} & x^{\frac{3}{2}}y^{-1} + x^{\frac{1}{2}}y^{-\frac{1}{3}} - x^{-\frac{1}{2}}y^{\frac{1}{3}} = \\
xy^{-\frac{2}{3}} - 2 + x^{-1}y^{\frac{2}{3}} & \quad \dfrac{x^{\frac{3}{2}}}{y} + \dfrac{x^{\frac{1}{2}}}{y^{\frac{1}{3}}} - \dfrac{y^{\frac{1}{3}}}{x^{\frac{1}{2}}}. \\
\underline{xy^{-\frac{2}{3}} - 1} & \\
-1 + x^{-1}y^{\frac{2}{3}} & \\
-1 + x^{-1}y^{\frac{2}{3}} &
\end{array}$$

3. Extract the square root of $9x-12x^{\frac{1}{2}}+10-4x^{-\frac{1}{2}}+x^{-1}$.

$$\begin{array}{l|r|l}
 & 9x\ -12x^{\frac{1}{2}}+10-4x^{-\frac{1}{2}}+x^{-1} & 3x^{\frac{1}{2}}-2+x^{-\frac{1}{2}}= \\
 & 9x \qquad\qquad\qquad\qquad\qquad & 3x^{\frac{1}{2}}-2+\dfrac{1}{x^{\frac{1}{2}}} \\
\hline
6x^{\frac{1}{2}}-2 & -12x^{\frac{1}{2}}+\ 4 \qquad\qquad & \\
\hline
 & 6-4x^{-\frac{1}{2}}+x^{-1} & \\
6x^{\frac{1}{2}}-4+x^{-\frac{1}{2}} & 6-4x^{-\frac{1}{2}}+x^{-1} &
\end{array}$$

EXERCISE XLV

Find the products of the following expressions:

1. $(x^{\frac{1}{4}}+y^{\frac{1}{4}})(x^{\frac{1}{4}}-y^{\frac{1}{4}})$.
2. $(x^{\frac{3}{4}}+3x)(x^{\frac{3}{4}}-3x)$.
3. $(a^{m-1}b+a^{m}b^{-2}-a^{n-2}b^{-1})ab$.
4. $(x+x^{-1}-1)(x-x^{-1}+1)$.
5. $(x^{\frac{1}{3}}-y^{\frac{2}{3}})(x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{2}{3}}+y^{\frac{4}{3}})$.
6. $(x^{\frac{1}{2}}+y^{\frac{1}{2}})(x^{\frac{3}{2}}-xy^{\frac{1}{2}}+x^{\frac{1}{2}}y-y^{\frac{3}{2}})$.
7. $(2a^{-2}+3a^{-1}-5)(2a^{-2}-3a^{-1}+5)$.
8. $(5x^{p-3}y^{r+3}-2x^{p-1}y^{r+1}-x^{p-2}y^{r+2})$
   $(3x^{p+4}y^{r-1}+4x^{p+5}y^{r-2}-x^{p+3}y^{r})$.

Find the quotients of the following expressions:

9. $(x^{2n}-y^{2n})\div(x^{n}+y^{n})$.
10. $(a^{-3n}-b^{6n})\div(a^{-n}-b^{2n})$.
11. $(x^{\frac{2}{3}}+x^{\frac{1}{3}}-6)\div(x^{\frac{1}{3}}-2)$.
12. $(x^{2}+x^{-2}-2)\div(x-x^{-1})$.
13. $(x^{\frac{3}{4}}-y^{\frac{3}{4}})\div(x^{\frac{1}{4}}-y^{\frac{1}{4}})$.
14. $(p^{\frac{1}{2}}-q)\div(p^{\frac{1}{10}}-q^{\frac{1}{5}})$.
15. $(a^{\frac{3}{4}}-b^{\frac{3}{4}})\div(a^{\frac{1}{2}}+a^{\frac{1}{4}}b^{\frac{1}{4}}+b^{\frac{1}{2}})$.
16. $(x^{3}-y^{2})\div(x^{\frac{1}{2}}+y^{\frac{1}{3}})$.
17. $(2a^{5}b^{-3}-5a^{4}b^{-2}+7a^{3}b^{-1}-5a^{2}+2ab)$
    $\div(a^{3}b^{-3}-a^{2}b^{-2}+ab^{-1})$.
18. $(\sqrt[3]{x^{4}}-4xy+4y\sqrt[3]{x^{2}}+4y^{2})\div(\sqrt[3]{x^{2}}+2x^{\frac{1}{3}}y^{\frac{1}{2}}+2y)$.

Extract the square root of the following expressions:

19. $x^{-8} - 6x^{-5} + 11x^{-2} - 6x + x^4.$

20. $4x^n + 9x^{-n} + 28 - 24x^{-\frac{n}{2}} - 16x^{\frac{n}{2}}.$

21. $1 + 4x^{-\frac{1}{3}} - 2x^{-\frac{2}{3}} - 4x^{-1} + 25x^{-\frac{4}{3}} - 24x^{-\frac{5}{3}} + 16x^{-2}.$

39. By the principles of the preceding articles many expressions may be simplified.

1. $$ab \cdot \frac{a^{-1} - b^{-1}}{(a-b)^{-1}} = ab \cdot (a-b)\left(\frac{1}{a} - \frac{1}{b}\right) = ab(a-b)\left(\frac{b-a}{ab}\right)$$
$$= -(a-b)^2.$$

2. $$\left(a^{n^2-1}\right)^{\frac{n}{n+1}} + \frac{\sqrt[n]{a^{2n}}}{a} = a^{n(n-1)} + \frac{a^2}{a} = a^{n^2-n} + a = \frac{a^{n^2}}{a^n} + a.$$

3. $$\frac{2^n \cdot (2^{n-1})^n}{2^{n+1} \cdot 2^{n-1}} = \frac{2^n \cdot 2^{n^2-n}}{2^{2n}} = \frac{2^{n^2}}{2^{2n}}.$$

EXERCISE XLVI

Simplify each of the following expressions, giving each result in a form free from radicals and from negative exponents:

1. $\left(\frac{4a^{-\frac{1}{4}}x^2}{9b^{-3}y^{\frac{1}{2}}}\right)^2.$

2. $\frac{\sqrt{a^{-\frac{5}{3}}b^3c^{-\frac{2}{3}}}}{\sqrt[3]{a^{\frac{1}{2}}b^4c^{-1}}}.$

3. $\left(\frac{8a^2}{27a^{-3}y^{\frac{1}{3}}}\right)^{-\frac{1}{3}}.$

4. $\frac{\sqrt{xa^{-1}}}{b^{\frac{1}{2}}y^{-\frac{3}{4}}} \div \frac{xa^{-\frac{3}{2}}}{b^2y^{-\frac{1}{2}}}.$

5. $\left[\frac{(-a)^{-\frac{5}{2}} \cdot a^{\frac{7}{2}}}{(-b)^2}\right]^2.$

6. $\left(\sqrt[b]{x^b} \div \sqrt[a]{x}\right)^{\frac{1}{1-a}}.$

7. $\left(64\sqrt[3]{36a^4b^{-2}}\right)^{\frac{1}{2}}.$

8. $\dfrac{a^{2n+1}b^{n-1}}{a^{n+2}b^{1-n}}$.

9. $\dfrac{4^{\frac{3}{4}} \cdot 3^{\frac{1}{2}} \cdot 2^{\frac{1}{3}}}{2^{\frac{3}{4}} \cdot 9^{\frac{1}{4}} \cdot 4^{\frac{1}{3}}}$.

10. $\dfrac{2x^2y^{-\frac{1}{2}}}{3a^2b^{-4}} \cdot \dfrac{6a^{-5}b^{\frac{1}{4}}}{7x^3y^{\frac{3}{4}}}$.

11. $(81a^4x^{-2}y^{\frac{2}{5}}z^{-\frac{1}{5}})^{\frac{1}{2}}$.

12. $a^{\frac{1}{2}}y^{\frac{1}{3}} \cdot \left(\dfrac{y^{\frac{1}{4}}}{x^{\frac{1}{6}}}\right)^2 \div \dfrac{y^{-\frac{1}{4}}}{x^{\frac{1}{4}}}$.

13. $\dfrac{x^{\frac{1}{3}}y^{\frac{1}{2}}}{x^{-\frac{2}{3}}} \div \dfrac{y^{-1}z^2}{x^{\frac{1}{6}}} \cdot \dfrac{z^3}{\sqrt{xy^{-\frac{3}{2}}}}$.

14. $(x^{q-r})^p \cdot (x^{r-p})^q \cdot (x^{p-q})^r$.

REVIEW EXERCISE XLVII

Simplify the following expressions:

1. $\sqrt[4]{49}$.
2. $\sqrt[5]{a^{10}}$.
3. a^0b^{-m}.
4. $\sqrt[6]{144}$.
5. $\sqrt[8]{256}$.
6. $\sqrt[3]{8x^6}$.
7. $\sqrt[3]{a^6x^{12}}$.
8. $\sqrt[3]{a^2b^6c^9}$.
9. $\sqrt{\dfrac{a+b}{a-b}}$.
10. $(3\sqrt[4]{\tfrac{1}{9}})^2$.
11. $\frac{1}{3}x^{\frac{2}{3}} \cdot 6x^{\frac{3}{2}}$.
12. $\sqrt{18} \cdot \sqrt{8}$.
13. $\sqrt{32a^{2m-1}b}$.
14. $\sqrt[3]{3a} \cdot \sqrt{2b}$.
15. $a^{\frac{3}{2}}$.
16. $\sqrt[10]{4}$.
17. $81^{\frac{3}{2}}$.
18. $\dfrac{a^a}{a^{-2}}$.
19. $\sqrt[3]{a^2}$.
20. $\dfrac{4\sqrt{6}}{3\sqrt{2}}$.
21. $(x^{\frac{m}{n}})^{\frac{p}{q}}$.
22. $(a+b)^0$.
23. $(2\sqrt{a^3})^2$.
24. $\sqrt{2} \cdot \sqrt[3]{3}$.
25. $\sqrt[4]{36a^2b^2}$.
26. $\sqrt{4a^2b^4c^8}$.
27. $(a^{2n+n^2})^{\frac{1}{3n}}$.
28. $(2ax^2y^{\frac{3}{2}})^2$.
29. $\sqrt{\dfrac{a^0b^4}{c^6d^2}}$.
30. $\dfrac{7\sqrt{48}}{3\sqrt{27}}$.
31. $\left(\dfrac{a^5}{a^{-3}}\right)^{\frac{1}{2}}$.
32. $\dfrac{a+x}{\sqrt{a+x}}$.
33. $\dfrac{\sqrt{50a^6b^4}}{\sqrt{2a^2b^2}}$.
34. $\sqrt{a^m} \cdot \sqrt[3]{b^n}$.
35. $\sqrt[3]{\dfrac{a}{bc}} \cdot \sqrt{\dfrac{a}{bd}}$.
36. $\sqrt{12} \cdot \sqrt[3]{72}$.
37. $\sqrt{108} - \sqrt{72}$.
38. $\sqrt[3]{54} + \sqrt[3]{128}$.

39. $\sqrt[3]{ab^{-1}c^{-2}} \cdot (a^{-1}b^{-2}c^{-4})^{-\frac{1}{6}}$.

40. $(2-\sqrt{-5})(3+4\sqrt{-5})$.

41. $x^{3p+q} \cdot x^{p-4r} \cdot (x^2)^{q-2r} \div x^{4p-8r}$.

42. $8^{-\frac{5}{3}} \cdot x^{\frac{1}{3}} \cdot \sqrt[3]{x\sqrt{9x^{-3}}} \div (64x^{-\frac{2}{3}})^{-\frac{1}{2}}$.

43. $\left\{x-\frac{1}{x}-3\left(x^{\frac{1}{3}}-\frac{1}{x^{\frac{1}{3}}}\right)\right\} \div (x^{\frac{1}{3}}-x^{-\frac{1}{3}})$.

44. $\left(\frac{a^{-\frac{1}{2}}}{4c^2}\right)^{-\frac{1}{2}}$.

45. $8^{-\frac{2}{3}}+16^{\frac{3}{4}}$.

46. $\left(\frac{64a^3}{27b^3}\right)^{-\frac{1}{3}}$.

47. $\sqrt{a^m} \cdot \sqrt{a^{2m}}$.

48. $2\sqrt[3]{6} \div 6\sqrt{2}$.

49. $\frac{2}{5}\sqrt{\frac{4}{5}} \cdot \frac{1}{3}\sqrt{\frac{1}{20}}$.

50. $(x^6)^{\frac{1}{3}}+4\sqrt[4]{x^8}$.

51. $\sqrt[3]{2} \cdot \sqrt[6]{\frac{1}{3}} \cdot \sqrt[3]{3}$.

52. $\sqrt[3]{2} \cdot \sqrt[6]{\frac{1}{8}} \cdot \sqrt[3]{2}$.

53. $\frac{3\sqrt{11}}{2\sqrt{98}} \cdot \frac{7\sqrt{22}}{5}$.

54. $(3\sqrt[n]{a^{-m}b^p})^{-mp}$.

55. $\sqrt{a+b} \cdot \sqrt{a-b}$.

56. $3\sqrt[6]{4a^2}+2\sqrt[3]{2a}$.

57. $\sqrt{24}+\sqrt{54}-\sqrt{6}$.

58. $2\sqrt{3}-\sqrt{12}+\sqrt[4]{9}$.

59. $\sqrt{20}+\sqrt{45}-\sqrt{\frac{4}{5}}$.

60. $\left(\frac{a^2b}{a^3b^{-4}}\right)^3 \div \left(\frac{ab^{-1}}{a^{-3}b^2}\right)^5$.

61. $\sqrt{75}+\sqrt{48}-\sqrt{243}$.

62. $(2+3\sqrt{5})(3\sqrt{5}-2)$.

63. $2abc\sqrt{20}+3a\sqrt{5b^2c^2}$.

64. $\frac{2}{\sqrt{5}+\sqrt{2}}$.

65. $\frac{\sqrt{7}+\sqrt{2}}{9+2\sqrt{14}}$.

66. $\sqrt{19-8\sqrt{3}}$.

67. $\sqrt{27-12\sqrt{5}}$.

68. $\sqrt[m]{a^2} \cdot b^3 \cdot \sqrt{c^4}$.

69. $\frac{\sqrt{-3}+3}{\sqrt{-4}-2\sqrt{3}}$.

70. $\dfrac{2\sqrt[3]{4}+5\sqrt[3]{32}}{\sqrt[3]{108}}.$

71. $\dfrac{a^{2m-3n}\cdot a^{-5m-n}}{a^{3m-4n}}.$

72. $\dfrac{a^{\frac{3}{5}}c^{-1}\sqrt{b}}{\sqrt[4]{b^3}}\cdot a^{-3}\sqrt[5]{ab^5}.$

73. $\sqrt{a^2-x^2}+\dfrac{x^2}{\sqrt{a^2-x^2}}.$

74. $2\sqrt{3}+3\sqrt{\frac{4}{3}}-5\sqrt{\frac{16}{3}}.$

75. $(6x^2-6)\left(\dfrac{3x-3}{2x+2}\right)^{\frac{1}{2}}.$

76. $4\sqrt{\frac{3}{4}}-\frac{2}{7}\sqrt{\frac{3}{16}}-2\sqrt{27}.$

77. $\dfrac{3\sqrt{1-x^2}-(1-x^2)^{-\frac{1}{2}}}{1-x^2}.$

78. $2\sqrt{24}\cdot 3\sqrt[4]{18}\cdot 4\sqrt[6]{24}.$

79. $\sqrt{a^{4n}b^{11}c^{2p}}\cdot\sqrt{a^nbc^{2-p}}.$

80. $\dfrac{x^4-2+x^{-4}}{x^2+2+x^{-2}}.$

81. $\dfrac{2\sqrt{3}-3}{1+\sqrt{3}-\sqrt{5}}.$

82. $\dfrac{\sqrt{-2}-3\sqrt{-6}}{\sqrt{-2}+2\sqrt{-5}}.$

83. $\dfrac{x+y^{-1}}{x^{\frac{1}{3}}+y^{-\frac{1}{3}}}-\dfrac{x-y^{-1}}{x^{\frac{1}{3}}-y^{-\frac{1}{3}}}.$

84. $\dfrac{\sqrt{3}}{2-\sqrt{3}}-\dfrac{2-\sqrt{-2}}{2+\sqrt{-2}}.$

85. $\dfrac{2x^2}{(1-x^2)^{\frac{3}{2}}}+\dfrac{1}{(1-x^2)^{\frac{1}{2}}}.$

86. $\sqrt{22+10\sqrt{-3}}.$

87. $(\sqrt[b]{x^b}\div\sqrt[a]{x^a})^{\frac{1}{1-a}}.$

88. $a^{\frac{1}{2}}y^{\frac{1}{3}}\cdot\left(\dfrac{y^{\frac{1}{4}}}{x^{\frac{1}{6}}}\right)^4\div\dfrac{y^{\frac{1}{4}}}{x^{\frac{1}{4}}}.$

89. $(a^3x^{-3}\div\sqrt[3]{a^2x^{-2}})^{\frac{2}{5}}.$

90. $\left(\dfrac{x^m}{y^n}\right)^{m-n}\div\left(\dfrac{y^{m+n}}{x^m}\right)^n.$

91. $\dfrac{1}{\sqrt{a}+\sqrt{b}}-\dfrac{1}{\sqrt{a}-\sqrt{b}}.$

92. $7\sqrt[3]{54}+\sqrt[6]{256}+\sqrt[3]{432}.$

93. $(81x-16)\div(3x^{\frac{1}{2}}+2).$

94. $(3\sqrt[5]{6}-2\sqrt{3})(\sqrt[3]{4}+4\sqrt[4]{9})$.

95. $(5\sqrt[4]{15}+2\sqrt[3]{9})(\sqrt[6]{6}+4\sqrt[12]{100})$.

96. $(3a^{-2}+\sqrt{2a^5}-\dfrac{1}{a}\sqrt{3a^3})^2$.

97. $\sqrt[5]{a\sqrt[3]{a^2}}\times\sqrt[6]{\sqrt[4]{a^9}}\times\sqrt[3]{a^2\sqrt[8]{a^7}}$.

98. $\dfrac{\sqrt{3+\sqrt{5}}-\sqrt{5-\sqrt{5}}}{\sqrt{3+\sqrt{5}}+\sqrt{5-\sqrt{5}}}$. **101.** $\sqrt[3]{x\sqrt{x^6}}\times\sqrt[3]{x\sqrt[3]{x^8}}$.

99. $(2\sqrt[3]{15}+3\sqrt[4]{10})^2$. **102.** $(2\sqrt{5}-10\sqrt{3})\div 2\sqrt{15}$.

100. $(5\sqrt[5]{20}-4\sqrt[3]{30})^2$. **103.** $\sqrt[10]{\frac{125}{16}}\times\sqrt[3]{\frac{4}{5}}$.

104. $\dfrac{5+2a}{2m-n}\sqrt{\dfrac{m^2-b^2}{y+x}}\times\dfrac{4m^2-n^2}{4a^2-25}\sqrt{\dfrac{x^2-y^2}{b+m}}$.

105. $(a^{\frac{1}{2}}-b^{\frac{1}{2}}-c^{\frac{1}{2}}+2\,b^{\frac{1}{4}}c^{\frac{1}{4}})\div(a^{\frac{1}{4}}+b^{\frac{1}{4}}-c^{\frac{1}{4}})$.

106. $(2a^{-\frac{5}{4}}-3a^{-\frac{3}{4}}+a^{-\frac{1}{4}}-2)\times(a^{-\frac{3}{4}}-2a^{-\frac{1}{4}}+3)$.

107. $\dfrac{a^{\frac{5}{3}}b^{\frac{1}{3}}-(ab)^{\frac{3}{2}}}{1-3\,a^{-\frac{1}{6}}b^{\frac{7}{6}}+3\,a^{-\frac{1}{3}}b^{\frac{7}{3}}-a^{-\frac{1}{2}}b^{\frac{7}{2}}}$.

108. $(9\,a^5-21\,a^3\sqrt{x}-\frac{25}{2}\,ax+12\,a^{-1}x^{\frac{3}{2}})\div(\frac{3}{2}\,a^3-4\,ax^{\frac{1}{2}})$.

109. Arrange in order of magnitude: $\sqrt{\frac{4}{3}}$, $\sqrt[3]{\frac{3}{2}}$, $\sqrt[4]{\frac{7}{4}}$.

110. Arrange in order of magnitude: $(\frac{1}{2})^{\frac{1}{2}}$ and $(\frac{2}{3})^{\frac{2}{3}}$.

111. Extract the square root of $x^4+4\,xy^2+y^{-2}+4\,x^{\frac{5}{2}}y$
$-2\,x^2y^{-1}-4\,x^{\frac{1}{2}}$.

112. $(a^{-4}-5\,b^4+4\,ab^5)\div(a^{-3}+2\,a^{-2}b+3\,a^{-1}b^2+4\,b^3)$.

CHAPTER VI

QUADRATIC EQUATIONS

40. An equation which contains, in its simplified form, the second power of the unknown quantity as the highest power of that unknown quantity is called a **quadratic equation.**

Thus, $\frac{x^3}{4}+\frac{x^2}{6}=\frac{1+3x^3}{12}+\frac{x}{2}$, which, when simplified, becomes $2x^2-6x-1=0$, is a quadratic equation.

41. Every quadratic equation may be reduced by the fundamental laws of algebra to the **general form,**

$$\boldsymbol{ax^2+bx+c=0,}$$

wherein a, b, and c are known quantities, and wherein $a \neq 0$. If $a=0$, the general form becomes $bx+c=0$, which is a simple equation.

If, in the general form, $b=0$, the resulting equation, $ax^2+c=0$, is called an **incomplete pure quadratic** equation.

If, in the general form, $c=0$, the resulting equation, $ax^2+bx=0$, is called an **incomplete quadratic** equation.

If, in the general form, neither $b=0$ nor $c=0$, the resulting equation, $ax^2+bx+c=0$, is called a **complete** (or **affected**) **quadratic** equation.

The known numbers, a, b, and c, are called the **coefficients** of the equation; and c is further called the **absolute** (or **constant**) **term.**

Thus, $x^2=4$, or $x^2-4=0$, is an incomplete pure quadratic equation in which $a=1$, $b=0$, $c=-4$; $3x^2+4x=0$ is an incomplete quadratic equation in which the coefficients are $a=3$, $b=4$, $c=0$; $4x^2+4x+3=0$ is a complete, or affected, quadratic equation in which $a=4$, $b=4$, $c=3$.

PURE QUADRATIC EQUATIONS

42. 1. Solve : $x^2=a^2$. (1)

Extracting the square roots in (1), $\pm x=\pm a$. (2)

The complete form of (2) is
$$\begin{cases} +x=+a, & (3)\\ -x=-a, & (4)\\ -x=+a, & (5)\\ \quad x=-a. & (6)\end{cases}$$

A value of $-x$ is not required; therefore,

multiplying (4) by -1, $x=\ \ a$, (7)

multiplying (5) by -1, $x=-a$. (8)

It is evident that (3) and (7) are identical; and that (6) and (8) are identical. Hence, *if the double sign be used only in the right member, the roots are not altered in value.* Thus,

Extracting the square roots in (1), $x=\pm a$.

VERIFICATION: $a^2=a^2$.

2. Solve: $\dfrac{x^2}{4}-20=\dfrac{x^2}{5}$. (1)

Clearing of fractions in (1), $5x^2-400=4x^2$, (2)

transposing and uniting in (2), $x^2=400$, (3)

extracting square roots in (3), $x=\pm 20$.

VERIFICATION: $\dfrac{400}{4}-20=\dfrac{400}{5}$.

NOTE. If x^2 is negative, the signs of all terms must be changed, since the square root of a negative number cannot be obtained.

EXERCISE XLVIII

Solve the following equations:

1. $x^2 = 169.$

2. $x^2 - a^2 = 0.$

3. $x^2 - 81 = 0.$

4. $3x^2 = 48.$

5. $25x^2 - b^2 = 0.$

6. $a^2x^2 = b^2x^2.$

7. $11x^2 = 36 + 2x^2.$

8. $x^2 = a^2 + 2ab + b^2.$

9. $ax^2 - ab = 2ax^2.$

10. $(7x)^2 = 296 - (5x)^2.$

11. $\dfrac{x+5}{x+13} = \dfrac{2x+7}{3x+18}.$

12. $\dfrac{2}{3} + \dfrac{x - \frac{3}{4}}{x^2} = -19\tfrac{1}{3} + \dfrac{x + \frac{1}{2}}{x^2}.$

13. $\dfrac{x + \frac{1}{2}}{x - \frac{1}{2}} + \dfrac{x - \frac{1}{2}}{x + \frac{1}{2}} = \dfrac{2}{x^2 - \frac{1}{4}}.$

14. $\dfrac{x+a}{x-a} + \dfrac{x-a}{x+a} = b.$

15. $\dfrac{ax+b}{cx+d} = \dfrac{cx-d}{ax-b}.$

16. $\dfrac{a(x-b)}{bx} = \dfrac{a}{b} - \dfrac{bx}{a}.$

17. $\dfrac{x-a^2}{x+a^2} + \dfrac{x+a^2}{x-a^2} = 3a^4 + 1.$

18. $\dfrac{12}{5}\left(x - 1 + \dfrac{2}{x+1}\right) = \dfrac{5}{x+1}.$

SOLUTION OF QUADRATIC EQUATIONS BY FACTORING

43. When the left member of a quadratic equation, reduced to the general form, can be factored, those values of x which reduce either factor to zero will satisfy the equation. Either factor may therefore be taken equal to zero, or **equated to zero**, and the roots of the factors are the roots of the equation.

The Factor Method holds for all forms of quadratic equations both complete and incomplete.

1. Solve by factoring: $9x^2 = 36$. (1)

Dividing (1) by 9, $x^2 = 4$, (2)

transposing in (2), $x^2 - 4 = 0$, (3)

factoring in (3), $(x+2)(x-2) = 0$, (4)

equating each factor in (4) to zero, $\begin{cases} x+2=0, \\ x-2=0, \end{cases}$ (5)

transposing in (5), $x = -2$, or $x = 2$. (6)

VERIFICATION: $9(-2)^2 = 36$; $9(2)^2 = 36$.

2. Solve by factoring: $ax^2 + bx = 0$. (1)

Factoring in (1), $x(ax+b) = 0$, (2)

equating each factor in (2) to zero, $\begin{cases} x=0, \\ ax+b=0, \end{cases}$ (3)

transposing in (3), $x = 0$, $ax = -b$, (4)

dividing $ax = -b$ by a, $x = -\frac{b}{a}$. (5)

VERIFICATION:

$$a(0) + b(0) = 0; \quad a\left(-\frac{b}{a}\right)^2 + b\left(-\frac{b}{a}\right) = \frac{b^2}{a} - \frac{b^2}{a} = 0.$$

3. Solve by factoring: $x^2 - 4x - 21 = 0$. (1)

Factoring in (1), $(x-7)(x+3) = 0$, (2)

equating each factor in (2) to zero, $\begin{cases} x-7=0, \\ x+3=0, \end{cases}$ (3)

transposing in (3), $x = 7$, or $x = -3$. (4)

VERIFICATION:

$$\begin{cases} (7)^2 - 4(7) - 21 = 49 - 28 - 21 = 0. \\ (-3)^2 - 4(-3) - 21 = 9 + 12 - 21 = 0. \end{cases}$$

EXERCISE XLIX

Solve the following equations by factoring:

1. $x^2+7x+12=0$.
2. $x^2+x-30=0$.
3. $x^2-x-12=0$.
4. $x^2+9x+20=0$.
5. $x^2+2x-224=0$.
6. $x^2-7x-260=0$.
7. $2x^2+9x-5=0$.
8. $3x^2-25x+28=0$.
9. $15x^2+23x-28=0$.
10. $-63x^2+16x-1=0$.
11. $x^2-\frac{9x}{20}+\frac{1}{20}=0$.
12. $40x^2-x-\frac{1}{20}=0$.
13. $x^2-(a+b)x+ab=0$.
14. $x^2-x(2p+5q)+10pq=0$.

NUMERICAL COMPLETE QUADRATIC EQUATIONS

44. If the coefficients of the equation $ax^2+bx+c=0$ are numerical, the equation is called a numerical complete quadratic equation.

Thus, $5x^2+7x-3=0$ is a numerical complete quadratic equation.

45. Solution by completing the Square. In $x^2 \pm 2nx + n^2$, the square of $x \pm n$, the third term is evidently the square of half the coefficient of x. If the left member of a complete quadratic equation contains the unknowns only, and the right member the absolute term, the equation may be made to assume the form $x^2 \pm 2nx$ by dividing the equation by the coefficient of x^2. The left member may be put into the form of the square of a binomial by adding the square of half the coefficient of x, a process which is called **completing the square.** This process is best understood by examples.

1. Solve the equation: $x^2 - 6x = 16$. (1)

The left member is already in the form $x^2 - 2nx$; that is, the coefficient of x^2 is unity. Half the coefficient of x is -3; $(-3)^2 = 9$. Therefore, adding 9 to *both* members of (1), so as not to destroy the equality, or,

completing the square in (1), $x^2 - 6x + 9 = 25$, (2)

extracting the square roots in (2), $x - 3 = \pm 5$, (3)

transposing and uniting in (4),

$$\left.\begin{aligned} x &= 3 + 5, \\ \text{or,} \quad x &= 3 - 5, \end{aligned}\right\} \quad (4)$$

$(x-3)^2 = 16$

combining in (4), $x = 8$, or $x = -2$. (5)

VERIFICATION:

$(x-3)^2 = 9+16$

$$(8)^2 - 6(8) = 16; \quad (-2)^2 - 6(-2) = 16.$$

$(x-3)^2 =$

2. Solve the equation: $x^2 - 14x - 11 = 0$. (1)

Transposing in (1), $x^2 - 14x = 11$, (2)

completing the square in (2), $x^2 - 14x + 49 = 60$, (3)

extracting the square roots in (3), $x - 7 = \pm 2\sqrt{15}$, (4)

transposing and uniting in (4),

$$\left.\begin{aligned} x &= 7 + 2\sqrt{15}, \\ \text{or,} \quad x &= 7 - 2\sqrt{15}. \end{aligned}\right\} \quad (5)$$

VERIFICATION:

$$\left\{\begin{aligned} &(7 + 2\sqrt{15})^2 - 14(7 + 2\sqrt{15}) - 11 = \\ &\quad 49 + 28\sqrt{15} + 60 - 98 - 28\sqrt{15} - 11 = 109 - 109 = 0. \\ &(7 - 2\sqrt{15})^2 - 14(7 - 2\sqrt{15}) - 11 = \\ &\quad 49 - 28\sqrt{15} + 60 - 98 + 28\sqrt{15} - 11 = 109 - 109 = 0. \end{aligned}\right.$$

3. Solve the equation : $x^2 - 3x = 4.$ (1)

Completing the square in (1), $x^2 - 3x + \frac{9}{4} = 4 + \frac{9}{4} = \frac{25}{4},$ (2)
extracting the square roots in (2), $x - \frac{3}{2} = \pm\frac{5}{2},$ (3)
transposing and uniting in (3), $x = 4$, or $x = -1.$ (4)

VERIFICATION : $(4)^2 - 3(4) = 4;\ (-1)^2 - 3(-1) = 4.$

4. Solve the equation : $\frac{1}{2x+1} + \frac{1}{3-x} = \frac{6}{5}.$ (1)

Clearing of fractions in (1),

$$5(3-x) + 5(2x+1) = 6(2x+1)(3-x), \tag{2}$$

simplifying in (2),

$$15 - 5x + 10x + 5 = -12x^2 + 30x + 18, \tag{3}$$

transposing and uniting in (3),

$$12x^2 - 25x = -2, \tag{4}$$

dividing (4) by 12, $x^2 - \frac{25x}{12} = -\frac{2}{12},$ (5)

completing the square in (5),

$$x^2 - \frac{25x}{12} + \left(\frac{25}{24}\right)^2 = \left(\frac{25}{24}\right)^2 - \frac{2}{12} = \frac{529}{576}, \tag{6}$$

extracting square roots in (6),

$$x - \tfrac{25}{24} = \pm\tfrac{23}{24}, \tag{7}$$

transposing and uniting in (7), $x = 2$, or $x = \frac{1}{12}.$ (8)

VERIFICATION :

$$\frac{1}{5} + 1 = \frac{6}{5};\ \frac{1}{\frac{1}{6}+1} + \frac{1}{3-\frac{1}{12}} = \frac{6}{7} + \frac{12}{35} = \frac{42}{35} = \frac{6}{5}.$$

Rule for solving Numerical Complete Quadratic Equations : *After clearing the equation of fractions (if any exist), trans-*

pose the unknowns to the left member and the absolute term to the right member; divide the equation by the coefficient of x^2; complete the square by adding to each member the square of half the coefficient of x; extract the square root of each member; solve the simple equations thus derived.

EXERCISE L

Solve the following equations by completing the square:

1. $x^2 + 6x = 55.$

2. $x^2 + 12x = 13.$

3. $x^2 + x - 2 = 0.$

4. $x^2 + x - 6 = 0.$

5. $8x^2 + 6x + 1 = 0.$

6. $x^2 + \frac{2x}{3} - \frac{8}{9} = 0.$

7. $x^2 + 2x + 40 = 0.$

8. $x^2 - 3x + 1 = 0.$

9. $x^2 + 5x - 7 = 0.$

10. $3x^2 + 5x = 2.$

11. $3x^2 - 7x = 16.$

12. $2x^2 - 5x + 3 = 0.$

13. $x(x + 1) = 12.$

14. $x + 3 = \frac{10}{x}.$

15. $x + \frac{1}{x} = 2.$

16. $\frac{2}{3x} + 4 = \frac{5x}{6x + 7}.$

17. $2x + 14 + \frac{7}{x} = 0.$

18. $x^2 - \frac{23x}{4} - 18 = 0.$

19. $\frac{1}{x^2} + \frac{5}{x} - 6 = 0.$

20. $11\frac{3}{4}x - 3\frac{1}{2}x^2 = -41\frac{1}{4}.$

21. $\frac{2}{x - 1} - \frac{1}{x + 3} = \frac{3}{8}.$

22. $\frac{5}{x + 2} + \frac{2}{x} = \frac{14}{x + 4}.$

23. $\frac{5}{4x^2 - 1} - \frac{3}{2x + 1} = \frac{2}{3}.$

24. $\frac{x + 3}{x + 2} + \frac{x - 3}{x - 2} = \frac{2x - 3}{x - 1}.$

25. $\frac{x^2 + 6}{x^2 - 4} + \frac{3}{2 - x} = 5 - \frac{7}{x + 2}.$

26. $9\frac{1}{3}x^2 - 90\frac{1}{3}x = -195.$

27. $\frac{17}{x} - \frac{32 - 11x}{3x^2} = 7\frac{2}{3}.$

LITERAL COMPLETE QUADRATIC EQUATIONS

46. If the coefficients of the equation $ax^2+bx+c=0$ are literal, the equation is called a literal complete quadratic equation.

Thus, $2ax^2+mx+6n=0$ is a literal complete quadratic equation.

The solution is found in the same manner as in the preceding paragraph.

1. Solve the equation: $x^2-bx-cx=(a+b)(a-c)$. (1)

Combining the terms in (1) which contain x,

$$x^2-x(b+c)=(a+b)(a-c), \tag{2}$$

completing the square in (2),

$$x^2-x(b+c)+\left(\frac{b+c}{2}\right)^2=\left(\frac{b+c}{2}\right)^2+(a+b)(a-c), \tag{3}$$

simplifying the right member in (3),

$$x^2-x(b+c)+\left(\frac{b+c}{2}\right)^2=\frac{4a^2+4ab-4ac+b^2-2bc+c^2}{4}, \tag{4}$$

extracting square roots in (4),

$$x-\frac{b+c}{2}=\pm\frac{2a+b-c}{2}, \tag{5}$$

transposing and uniting in (5),

$$x=a+b, \text{ or } x=c-a. \tag{6}$$

VERIFICATION: $$\left\{\begin{aligned}(a+b)^2-(a+b)(b+c)&=(a+b)(a-c),\\ (a+b)(a-c)&=(a+b)(a-c).\\ (c-a)^2-(c-a)(b+c)&=(a+b)(a-c),\\ (c-a)(-a-b)&=(a+b)(a-c).\end{aligned}\right.$$

2. Solve the equation: $ax^2 + bc - bx = acx.$ (1)

Transposing in (1), $ax^2 - bx - acx = -bc,$ (2)

combining in (2), $ax^2 - x(b + ac) = -bc,$ (3)

dividing (3) by a, $x^2 - x\left(\frac{b+ac}{a}\right) = -\frac{bc}{a},$ (4)

completing the square in (4),

$$x^2 - x\left(\frac{b+ac}{a}\right) + \left(\frac{b+ac}{2a}\right)^2 = \left(\frac{b+ac}{2a}\right)^2 - \frac{bc}{a}, \tag{5}$$

simplifying the right member in (5),

$$x^2 - x\left(\frac{b+ac}{a}\right) + \left(\frac{b+ac}{2a}\right)^2 = \frac{b^2 - 2abc + a^2c^2}{4a^2}, \tag{6}$$

extracting square roots in (6),

$$x - \frac{b+ac}{2a} = \pm\frac{b-ac}{2a}, \tag{7}$$

transposing and uniting in (7), $x = \frac{b}{a}$, or $x = c.$ (8)

Verification:

$$\begin{cases} a\left(\frac{b^2}{a^2}\right) + bc - b\left(\frac{b}{a}\right) = ac\left(\frac{b}{a}\right), \quad \frac{b^2}{a} + bc - \frac{b^2}{a} = bc. \\ ac^2 + bc - bc = ac^2. \end{cases}$$

The left member should always be written so as to show the coefficients of x^2 and of x.

EXERCISE LI

Solve the following equations by completing the square:

1. $x^2 + 4bx = -4b^2.$
2. $x^2 - 5ax + 6a^2 = 0.$
3. $x^2 + ax - 2a^2 = 0.$
4. $9x^2 - 6pq = 2pq - 3qx.$
5. $bx^2 + ac = (a + bc)x.$
6. $x^2(a + + b)x + ab = 0.$

7. $x^2 + ax = a^2.$

8. $x + \frac{1}{x} = a + \frac{1}{a}.$

9. $lx^2 + mx + n = 0.$

10. $x^2 - 2\,ax + b = 0.$

11. $\frac{x-a}{x+a} - \frac{x+a}{x-a} = 5.$

12. $ax^2 - 2\,bx + c = 0.$

13. $x^2 - 4\,ax + a^2 = 0.$

14. $\frac{x}{a^2b(x+c)} = \frac{x+c}{d^2bc}.$

15. $ax^2 + a = (a^2+1)x.$

16. $x^2 - 2\,ax + a^2 - b^2 = 0.$

17. $\frac{1}{a-x} - \frac{1}{a+x} = \frac{3+x^2}{a^2-x^2}.$

18. $mqx^2 - mnx + pqx = np.$

19. $\frac{(4\,a^2-b^2)(x^2+1)}{4\,a^2+b^2} = 2\,x.$

20. $\frac{a}{x-b} + \frac{b}{x-a} = 2.$

21. $x(1-x) = ax^2 + b.$

22. $1 - 7\,x^2 = 2\,ax - bx^2.$

23. $\frac{(x-a)^2}{a(x+a)} = \frac{b - \frac{2\,c}{x+a}}{b - \frac{c}{x-a}}.$

24. $\frac{x}{a} + \frac{1}{a} \cdot \frac{1+a}{x} = x + \frac{1}{a^2}.$

25. $a^2x - 2\,b^2 = ab \cdot \frac{x^2+1}{x+1}.$

26. $\frac{a}{b+x} + \frac{b}{a+x} = \frac{2\,a}{2\,a-b}.$

27. $\frac{x-a}{2\,b} - \frac{x-2\,b}{x-b} = \frac{b}{a+b}.$

28. $a^2 \cdot \frac{2\,x-1}{x+2} = b^2 \cdot \frac{x+2}{2\,x-1}.$

29. $\frac{a+x}{a-x} - \frac{a-x}{a+x} = \frac{4\,b(a+b)}{a(a+2\,b)}.$

30. $\frac{3\,x}{a-b+2\,x} = \frac{5\,x-5\,a+7\,b}{a+b-x}.$

31. $cx^2 - (a+b+c)x + (a+b) = 0.$

32. $mnx^2 - (m+n)(mn+1)x + (m+n)^2 = 0.$

33. $2\,x^2(a^2-b^2) - (3\,a^2+b^2)(x-1) = (3\,b^2+a^2)(x+1).$

34. $\dfrac{x-2a}{b+x}+\dfrac{x-b}{x-a}=\dfrac{b+4a}{2(a+b)}$

35. $x^2(a+b)^2-x(a^2-b^2)=ab.$

36. $\dfrac{2(a+b)}{x-b}+\dfrac{2b}{x-a}=\dfrac{3(a-b)}{x-3b}.$

37. $\dfrac{x-a}{x+3b}+\dfrac{x}{x-b}=\dfrac{2(x+a)}{x+3a-4b}.$

38. $\dfrac{a(1-x^2)}{bx}+\dfrac{(2a-b)x}{a}=\dfrac{4a}{a+b}.$

39. $\dfrac{a}{b}(x+1)+\dfrac{b}{a}(x-1)=\dfrac{2x^2-1}{x}.$

40. $(a-x)\left(1-\dfrac{3a+3x}{c-\frac{1}{2}}\right)-2=\dfrac{1-2a}{1-2c}(c-3)-\dfrac{1+a}{1-\frac{1}{2}c}.$

41. $\dfrac{x+b}{2a}+\dfrac{2a}{x-b}=1-\dfrac{2a}{b}\left(1-\dfrac{2a-b}{x-b}\right).$

42. $(a^2+b^2)(4x^2+1)+2ab(4x^2-1)=4x(a^2-b^2).$

43. $ax-\dfrac{4}{ax+b}[b^2(1+x)x-a^2(1-x)]=b.$

44. $(a+1)\dfrac{x-2}{x-1}-\dfrac{a}{a+1}\left(2+\dfrac{1}{x}-\dfrac{(a+1)^2}{a(x-1)}\right)=0.$

45. $\dfrac{2a(a+b)-b^2x}{bx-2a}=\dfrac{2}{\dfrac{b}{a^2}\left(\dfrac{1}{x}-\dfrac{b}{2a}\right)}-b\left\{1-\dfrac{b}{a-\dfrac{b}{2x}}\right\}.$

46. $\dfrac{2}{\dfrac{1}{b}\left(\dfrac{x}{2a}-2\right)}-\dfrac{b}{a}\left[\dfrac{4a}{\dfrac{4b}{x}-\dfrac{b}{a}}-\dfrac{1}{\dfrac{1}{x}-\dfrac{1}{b}}\right]=0.$

SOLUTION OF QUADRATIC EQUATIONS BY A FORMULA

47. Every quadratic equation may be reduced to the general form, $ax^2+bx+c=0$.

Solve the equation: $ax^2+bx+c=0.$ (1)

Transposing in (1), $ax^2+bx=-c,$ (2)

dividing (2) by a, $x^2+x\left(\frac{b}{a}\right)=-\frac{c}{a},$ (3)

completing the square in (3),

$$x^2+x\left(\frac{b}{a}\right)+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}, \tag{4}$$

simplifying the right member in (4),

$$x^2+x\left(\frac{b}{a}\right)+\frac{b^2}{4a^2}=\frac{b^2-4ac}{4a^2}, \tag{5}$$

extracting square roots in (5),

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}, \tag{6}$$

transposing and uniting in (6), $x=\frac{-b+\sqrt{b^2-4ac}}{2a}$

or, $x=\frac{-b-\sqrt{b^2-4ac}}{2a}.$ (7)

The values of x in (7) are general values. The values of the roots in any particular equation are found by substituting in the formulas in (7) the values of a, b, and c in any particular equation.

1. Solve by the formula: $2x^2-5x=3.$ (1)

Putting (1) in the general form, $2x^2-5x-3=0.$ (2)

In (2), $a=2,\ b=-5,\ c=-3.$ (3)

Substitute for a, b, and c their values from (3) in the formulas in (7),

$$\left.\begin{aligned} x &= \frac{5+\sqrt{(-5)^2-4(2)(-3)}}{2(2)} = 3, \\ \text{or,}\quad x &= \frac{5-\sqrt{(-5)^2-4(2)(-3)}}{2(2)} = -\frac{1}{2}. \end{aligned}\right\} \tag{4}$$

VERIFICATION: $\begin{cases} 2(3)^2-5(3)-3=18-15-3=0, \\ 2(-\frac{1}{2})^2-5(-\frac{1}{2})-3=\frac{1}{2}+\frac{5}{2}-3=0. \end{cases}$

The formulas are written in the more compact form,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

EXERCISE LII

Solve the following equations by the formula:

1. $x^2-10x+25=0.$
2. $6x^2+13x-8=0.$
3. $5x^2+11x+83=0.$
4. $x(x-2)=5(x-6)^2.$
5. $x^2+\frac{3x}{7}=10\frac{3}{4}.$
6. $x(x+1)=\frac{103}{36}.$
7. $x^2-2ax+b=0.$
8. $ax^2-2bx+c=0.$
9. $ax^2-(a+b)x+b=0.$
10. $\frac{2x(2x-5)}{2x-1}-\frac{2}{2x-1}=3.$
11. $(a+b)bx^2+a^2=a(a+2b)x.$
12. $(x-2)^2+(x+5)^2=(x+6)^2.$
13. $(b^2-1)x^2-2(ab-1)x+a^2=1.$
14. $(a-1)x^2+(a+1)x+\frac{a}{a-1}=0.$

IRRATIONAL QUADRATIC EQUATIONS

48. Quadratic equations which contain indicated roots of the unknown quantities are called **irrational**, or **radical quadratic equations.**

Thus, $\sqrt{x+3}+x=9$, is an irrational quadratic equation.

Roots obtained in solving quadratic equations involving radicals must be substituted in the original equation for the purpose of verification.

1. Solve the equation: $\sqrt{x+7}=x+1.$ (1)

Squaring (1), $x+7=x^2+2x+1,$ (2)

transposing and uniting in (2), $x^2+x-6=0,$ (3)

solving (3), $x=2$, or $x=-3.$ (4)

VERIFICATION: $\begin{cases}\sqrt{2+7}=2+1; \text{ 2 is a root of (1).}\\ \sqrt{-3+7}\neq -3+1; \text{ -3 is not a root of (1).}\end{cases}$

2. Solve the equation: $\sqrt{2x+21}-\sqrt{x+7}=2.$ (1)

Squaring (1),

$$2x+21-2\sqrt{2x^2+35x+147}+x+7=4, \quad (2)$$

transposing and uniting in (2),

$$-2\sqrt{2x^2+35x+147}=-24-3x, \quad (3)$$

squaring (3), $8x^2+140x+588=576+144x+9x^2,$ (4)

transposing and uniting in (4),

$$x^2+4x-12=0, \quad (5)$$

solving (5), $x=2$, or $x=-6.$ (6)

VERIFICATION: $\begin{cases}\sqrt{4+21}-\sqrt{2+7}=2; \text{ 2 is a root of (1).}\\ \sqrt{-12+21}-\sqrt{-6+7}=2; \text{ -6 is a root of (1).}\end{cases}$

EXERCISE LIII

Solve the following equations and verify the roots:

1. $\frac{1}{2}\sqrt{x}+1=3x.$

2. $2x-\sqrt{x+3}=-5.$

3. $\sqrt{x+5}+x=7.$

4. $4x-\sqrt{x+3}=x-5.$

5. $9x-\sqrt{9x+1}=2x-1.$

6. $\dfrac{5+\sqrt{x}}{3\sqrt{x}+1}+\sqrt{x}=3.$

7. $\dfrac{\sqrt{x+1}-2}{2\sqrt{x}-1}=2\sqrt{x}+1.$

8. $\sqrt{\dfrac{9x-2}{x-1}}-3=\dfrac{2}{x}.$

9. $\sqrt{x+1+\sqrt{5(x+2)}}=3.$

10. $\sqrt{2x-7}+\sqrt{7x+8}=11.$

11. $\sqrt{3x+4}+\sqrt{5(x+1)}=9.$

12. $\sqrt{a+x}+\sqrt{b-x}=\sqrt{a+b}.$

13. $2\sqrt{3x+1}-3\sqrt{x+3}+2=0.$

14. $3\sqrt{3x-4}+4x=10(x-1).$

15. $\dfrac{\sqrt{5x^2-2x+1}}{2\sqrt{x}}=\sqrt{4x-3}.$

16. $3\sqrt{x+8}-\sqrt{x-8}=2\sqrt{2x+2}.$

17. $3\sqrt{3x+1}-2\sqrt{x+3}=\sqrt{2(x+1)}.$

18. $\sqrt{4x+1}+3\sqrt{9x-2}=5\sqrt{5x-1}.$

19. $\dfrac{x}{\sqrt{x}+\sqrt{a-x}}+\dfrac{x}{\sqrt{x}-\sqrt{a-x}}=\dfrac{8a}{3\sqrt{x}}.$

SOLUTIONS OF EQUATIONS IN THE QUADRATIC FORM

49. An equation which contains only two different powers of the unknown quantity one of which is double the other is said to be in the **quadratic form.** The general type of equations in the quadratic form is $\boldsymbol{ax^{2n}+bx^n+c=0}$. Equations in the quadratic form may be solved like quadratics.

1. Solve the equation: $x^4-2x^2+1=0. \qquad (1)$

Writing (1) in the quadratic form,

$$(x^2)^2-2(x^2)+1=0, \qquad (2)$$

factoring (2),
$$(x^2-1)(x^2-1)=0, \qquad (3)$$

equating the factors in (3) to zero,

$$\begin{cases} x^2-1=0, \\ x^2-1=0, \end{cases} \qquad (4)$$

transposing in (4),
$$x^2=1, \qquad (5)$$

extracting square roots in (5),
$$x=\pm 1. \qquad (6)$$

VERIFICATION: $1-2+1=0$.

2. Solve the equation: $x^{\frac{3}{2}}-9x^{\frac{3}{4}}+8=0. \qquad (1)$

Writing (1) in the quadratic form,

$$(x^{\frac{3}{4}})^2-9(x^{\frac{3}{4}})+8=0, \qquad (2)$$

factoring in (2),
$$(x^{\frac{3}{4}}-8)(x^{\frac{3}{4}}-1)=0, \qquad (3)$$

equating the factors in (3) to zero,

$$\begin{cases} x^{\frac{3}{4}}-8=0, \\ x^{\frac{3}{4}}-1=0, \end{cases} \qquad (4)$$

transposing in (4),
$$x^{\frac{3}{4}}=8,\ x^{\frac{3}{4}}=1, \qquad (5)$$

raising each equation in (5) to $\frac{4}{3}$ power,

$$(x^{\frac{3}{4}})^{\frac{4}{3}} = 8^{\frac{4}{3}},\ (x^{\frac{3}{4}})^{\frac{4}{3}} = (1)^{\frac{4}{3}}, \qquad (6)$$

simplifying in (6), $x = 16$, or $x = 1$. (7)

VERIFICATION: $\begin{cases} (16)^{\frac{3}{2}} - 9\,(16)^{\frac{3}{4}} + 8 = 64 - 72 + 8 = 0. \\ 1 - 9 + 8 = 0. \end{cases}$

3. Solve the equation :

$$x^2 - 7\,x - \sqrt{x^2 - 7\,x + 18} = 12. \qquad (1)$$

Adding 18 to each member in (1),

$$(x^2 - 7\,x + 18) - \sqrt{x^2 - 7\,x + 18} = 30, \qquad (2)$$

writing (2) in the quadratic form,

$$(\sqrt{x^2 - 7\,x + 18})^2 - \sqrt{x^2 - 7\,x + 18} = 30, \qquad (3)$$

transposing in (3),

$$(\sqrt{x^2 - 7\,x + 18})^2 - \sqrt{x^2 - 7\,x + 18} - 30 = 0, \qquad (4)$$

factoring in (4),

$$(\sqrt{x^2 - 7\,x + 18} - 6)\,(\sqrt{x^2 - 7\,x + 18} + 5) = 0, \qquad (5)$$

equating the factors in (5) to zero,

$$\begin{cases} \sqrt{x^2 - 7\,x + 18} - 6 = 0, \\ \sqrt{x^2 - 7\,x + 18} + 5 = 0, \end{cases} \qquad (6)$$

transposing in (6),

$$\sqrt{x^2 - 7\,x + 18} = 6,\ \sqrt{x^2 - 7x + 18} = -5. \qquad (7)$$

Solving $\sqrt{x^2 - 7\,x + 18} = 6$, $x = 9$, or -2.

$\sqrt{x^2 - 7\,x + 18} = -5$ is impossible since the radical cannot equal a negative quantity.

VERIFICATION: on substitution in (1), both 9 and -2 are roots.

EXERCISE LIV

Solve the following equations:

1. $x - \sqrt{x} = 2.$
2. $x^4 - 5x^2 - 126 = 0.$
3. $x^4 - 30x^2 + 125 = 0.$
4. $x^{\frac{4}{3}} + 5x^{\frac{2}{3}} = 36.$
5. $x^{\frac{2}{3}} - 6x^{\frac{1}{3}} = 16.$
6. $2x^{-\frac{2}{3}} - x^{-\frac{1}{3}} - 6 = 0.$
7. $x^{-3} - x^{-\frac{3}{2}} = -\dfrac{7}{64}.$
8. $\left(x + \dfrac{8}{x}\right)^2 - x - 72 = \dfrac{8}{x}.$
9. $\dfrac{55}{(x+7)^2} - \dfrac{4}{(x+7)} = \dfrac{1}{11}.$
10. $\sqrt{x^2 - a^2} = 2 - \dfrac{1}{\sqrt{x^2 - a^2}}.$
11. $\sqrt{x^2 + 12} + \sqrt[4]{x^2 + 12} = 6.$
12. $x + \sqrt{x^2 - ax + b^2} = \dfrac{x^2}{a} + b.$
13. $\dfrac{120}{x^2 + 8x + 16} + \dfrac{11}{x + 4} = 17.$
14. $\sqrt[4]{x - 1} + 2\sqrt{x - 1} - 1 = 0.$
15. $x^2 - 2\sqrt{x^2 + 4x - 5} = 13 - 4x.$
16. $49x^2 + 42x + 9 = 1 - (7x + 3).$
17. $3x^2 + 15x - 2\sqrt{x^2 + 5x + 1} = 2.$
18. $2x^2 + 3x - 5\sqrt{2x^2 + 3x + 9} = -3.$
19. $\dfrac{3}{(x^2 - 5x + 7)^2} - \dfrac{2}{x^2 - 5x + 7} + \dfrac{1}{3} = 0.$
20. $4x^2 + 22x - 3\sqrt{2x^2 + 11x + 13} = 78.$
21. $\sqrt[6]{x^2 + 5x + 28} + \sqrt[3]{x^2 + 5x + 28} - 6 = 0.$

CHARACTER OF THE ROOTS

50. The roots of the general form $ax^2+bx+c=0$ have been found, § 47, to be: $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. Upon the nature of $\sqrt{b^2-4ac}$ will depend the character of the roots. The quantity, b^2-4ac, is called the **discriminant.**

(1) **If b^2-4ac is positive,** that is, if $b^2-4ac>0$, the roots are real and unequal, and either (*a*) rational or (*b*) irrational. If the discriminant is (*a*) a perfect square, the roots are real and rational; if (*b*) not a perfect square, the roots are real and irrational.

Thus, in the equation $6x^2+5x-21=0$, since $a=6$, $b=5$, $c=-21$, the discriminant is $529=23^2$. Therefore, the roots are real, rational, and unequal.

In the equation $2x^2+5x-4=0$, since $a=2$, $b=5$, $c=-4$, the discriminant is 57. Therefore, the roots are real, irrational, and unequal.

(2) **If b^2-4ac is zero,** that is, if $b^2=4ac$, the roots are real, rational, and equal.

Thus, in the equation $4x^2-12x+9=0$, since $a=4$, $b=-12$, $c=9$, the discriminant is 0. Therefore, the roots are real, rational, and equal.

(3) **If b^2-4ac is negative,** that is, if $b^2-4ac<0$, the roots are imaginary and unequal.

Thus, in the equation $x^2-2x+4=0$, since $a=1$, $b=-2$, and $c=4$, the discriminant is -12. Therefore, the roots are imaginary and unequal.

The character of the roots of any given equation may therefore be found by evaluating the discriminant.

The following summary will be found useful:

(1) **If $b^2 - 4ac > 0$, the roots are real and unequal.**

(2) **If $b^2 = 4ac$, the roots are real and equal.**

(3) **If $b^2 - 4ac < 0$, the roots are imaginary and unequal.**

1. Determine, without solving, the character of the roots of $2x^2 - 7x + 5 = 0$.

$$a = 2,\ b = -7,\ c = 5.$$

$$b^2 - 4ac = 49 - 4(2)(5) = 9 = 3^2.$$

Roots are real, rational, and unequal.

2. Determine, without solving, the character of the roots of $9x^2 - 12x + 4 = 0$.

$$a = 9,\ b = -12,\ c = 4.$$

$$b^2 - 4ac = 144 - 4(9)(4) = 0.$$

Roots are real and equal.

3. Determine, without solving, the character of the roots of $4x^2 - 4x + 5 = 0$.

$$a = 4,\ b = -4,\ c = 5.$$

$$b^2 - 4ac = 16 - 4(4)(5) = -64.$$

Roots are imaginary and unequal.

4. For what value of m are the roots equal in the equation $3x^2 + 4x + m = 0$?

$$a = 3,\ b = 4,\ c = m.$$

If the roots are equal,
$$b^2 - 4ac = 0,$$
$$16 - 4(3)m = 0,$$
$$16 - 12m = 0,$$
$$12m = 16,$$
$$m = \tfrac{4}{3}.$$

EXERCISE LV

Determine by the use of the discriminant the character of the roots in the following equations:

1. $x^2 - 4x + 4 = 0.$
2. $x^2 - 5x + 6 = 0.$
3. $x^2 - 2x - 1 = 0.$
4. $2x^2 - 3x + 5 = 0.$
5. $5x^2 - 2x + 1 = 0.$
6. $x^2 - 3x + 1 = 0.$
7. $x^2 - 4x + 7 = 0.$
8. $x^2 - 7x + 12 = 0.$
9. $3x^2 - 4x + 1 = 0.$
10. $2x^2 - 13x + 5 = 0.$
11. $3x^2 - 4x + 12 = 0.$
12. $2x^2 - 5x - 5 = 0.$
13. $3x^2 - 5x = 2.$
14. $x^2 - 2ax = (b + a)(b - a).$

Determine the value of m for which the roots are equal in the following equations:

15. $2x^2 + 4x + m = 0.$
16. $mx^2 + 6x + 3 = 0.$
17. $3x^2 + 4x - m = 0.$
18. $16x^2 + 8mx + 1 = 0.$
19. $4x^2 - 12x + m = 0.$
20. $mx^2 - (8 + m)x + 9 = 0.$

RELATION BETWEEN ROOTS AND COEFFICIENTS

51. It is convenient to derive the formula for the general equation $ax^2 + bx + c = 0$, where the coefficient of x^2 is unity. Dividing the general equation by a, $x^2 + \frac{bx}{a} + \frac{c}{a} = 0$; in the last equation, putting $p = \frac{b}{a}$, and $q = \frac{c}{a}$, the equation is $\boldsymbol{x^2 + px + q = 0}$.

The roots of $x^2 + px + q = 0$ are found to be

$$x = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

Let $$\alpha = \frac{-p+\sqrt{p^2-4q}}{2}, \tag{1}$$

and $$\beta = \frac{-p-\sqrt{p^2-4q}}{2}, \tag{2}$$

adding (1) and (2), $\alpha+\beta=-p,$ (3)

multiplying (1) and (2), $\alpha\beta=q.$ (4)

Hence, in the equation $x^2+px+q=0$:

(1) *The sum of the roots equals the coefficient of x with its sign changed.*

(2) *The product of the roots equals the absolute term.*

52. Since the equation $x^2+px+q=0$ is the general form of complete quadratics, the sum and the product of the roots of any complete quadratic may be found by inspection.

1. Find by inspection the sum and product of the roots of $2x^2+3x+1=0.$ (1)

Dividing (1) by 2, $x^2+\frac{3}{2}x+\frac{1}{2}=0,$ (2)

if α and β are the roots, by the rule, $\alpha+\beta=-\frac{3}{2},$ (3)

$\alpha\beta=\frac{1}{2}.$ (4)

The equation $x^2+px+q=0$, whose roots are α and β (read, respectively, "alpha" and "beta"), may be written $\boldsymbol{x^2-(\alpha+\beta)x+\alpha\beta=0}$.

2. Form an equation whose roots are -2 and 3.

Take $\alpha=-2$, and $\beta=3$.

Then, $\alpha+\beta=-2+3=1$; $\alpha\beta=(-2)(3)=-6.$

Substituting for $\alpha+\beta$ and $\alpha\beta$ these values in $x^2-(\alpha+\beta)x+\alpha\beta=0,$

$$x^2-(1)x-6=0, \text{ or, } x^2-x-6=0.$$

3. Form an equation whose roots shall be the squares of the roots of the equation $x^2+px+q=0$. (1)

Let α and β be the roots of (1).

By the conditions the required equation is,

$$x^2-(\alpha^2+\beta^2)x+\alpha^2\beta^2=0. \qquad (2)$$

Now $\alpha\beta=q$; hence, $\alpha^2\beta^2=q^2$.

Again, $\alpha+\beta=-p$; hence, $\alpha^2+2\,\alpha\beta+\beta^2=p^2$,

$$\alpha^2+\beta^2=p^2-2\,\alpha\beta,$$

$$\alpha^2+\beta^2=p^2-2\,q.$$

Substituting $\alpha^2+\beta^2=p^2-2\,q$ and $\alpha^2\beta^2=q^2$, in (2),

$$x^2-(p^2-2\,q)x+q^2=0.$$

4. Form an equation whose roots are reciprocals of the roots of the equation $x^2+px+q=0$. (1)

Let α and β be the roots of (1).

By the conditions, the required equation is,

$$x^2-\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)x+\frac{1}{\alpha\beta}=0, \text{ or, } x^2-\left(\frac{\alpha+\beta}{\alpha\beta}\right)x+\frac{1}{\alpha\beta}=0. \qquad (2)$$

Now, $\alpha+\beta=-p$; and $\alpha\beta=q$.

Substituting $\alpha+\beta=-p$, and $\alpha\beta=q$, in (2),

$$x^2-\left(\frac{-p}{q}\right)x+\frac{1}{q}=0, \qquad (3)$$

$$x^2+\frac{p}{q}x+\frac{1}{q}=0,$$

$$qx^2+px+1=0.$$

The results obtained in the preceding examples may be verified by solving the equation.

EXERCISE LVI

Form that equation whose roots are respectively:

1. 2 and 3.

2. 5 and 2.

3. 6 and -2.

4. -3 and -5.

5. $\frac{1}{2}$ and $\frac{1}{3}$.

6. 4 and $-\frac{3}{5}$.

7. -5 and $\frac{2}{7}$.

8. -4 and $\frac{1}{4}$.

9. $2+\sqrt{3}$ and $2-\sqrt{3}$.

10. $\frac{1+\sqrt{3}}{2}$ and $\frac{1-\sqrt{3}}{2}$.

11. $-\frac{1-2\sqrt{5}}{3}$ and $-\frac{1+2\sqrt{5}}{3}$.

12. $1+2\sqrt{-3}$ and $1-2\sqrt{-3}$.

13. $2+\sqrt{-2}$ and $2-\sqrt{-2}$.

14. $a+\sqrt{b}$ and $a-\sqrt{b}$.

15. $a+\sqrt{-b}$ and $a-\sqrt{-b}$.

16. $-c+\sqrt{-d}$ and $-c-\sqrt{-d}$.

17. Form a quadratic equation whose second member shall be 0, whose absolute term in the first member shall be -4, and one of whose roots shall be $-\frac{1}{3}$.

18. One root of the equation $4x^2-16x+4=0$ is $2+\sqrt{3}$; find the second root.

19. Find, without solving, the sum and product of the roots of the equation $3x^2-7x-5=0$.

20. Form an equation whose roots shall be the reciprocals of the roots of the equation $2x^2-x+1=0$.

21. Form an equation whose roots shall have the same absolute value as, but signs opposite to, the roots of $x^2+px+q=0$.

CHAPTER VII

GRAPHS

53. All real numbers may be pictured by points on a line of indefinite length. Choose any point *O* on the line as the center and a fixed distance which will represent unity. Lay off from this point the proper number of these distances to the right to represent positive numbers and to the left to represent negative numbers. The points so located may be used to picture to the eye the corresponding numbers. In this way every positive and negative integer and fraction may be represented by a point which is different from the point which represents any other number.

In § 10 it was shown that certain irrational numbers may also be represented by points on a line, and we shall assume that every irrational number may be so represented. We shall also assume that to every point on the line there corresponds either a rational or an irrational number.

The student is probably familiar with drawings in which changes in temperature or in the amount of rainfall, or the rise and fall of some commodity, is represented by a curve. In the present chapter it will be shown how the relation between two algebraic quantities, which are connected in any way, can be represented by drawings to scale. As a first step in this direction, it is necessary to establish certain conventions, by the aid of which the position of any point in a single flat surface, or plane, can be fixed by two algebraic quantities.

54. Constructing a pair of perpendicular lines, called **axes**, $X'X$ and $Y'Y$, as shown in Fig. 2, a point can be located by saying that it is m units above or below $X'X$, and n units to the right or left of $Y'Y$.

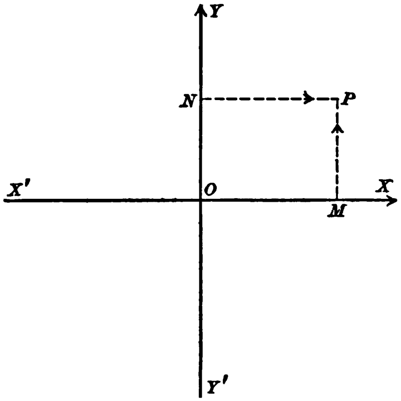


FIG. 2.

If, instead of using the words "above" and "below," "right" or "left," it is understood that *all distances measured upward or to the right are positive, and those measured downward or to the left are negative, two numbers with the proper signs attached will represent the distances of the point from the two lines, and these two numbers taken together will locate absolutely the position of any point in the same plane with the lines.*

55. The distance NP of a point to the right or left of $Y'Y$ is represented by x and is given first; the distance MP of this point above or below $X'X$ is represented by y and is given second. These two distances are called the **coördinates** of the point. The coördinates are written in parenthesis; thus, $P = (3, 4)$ means that the point P is 3 units to the right of the vertical line $Y'Y$, and 4 units above the horizontal line $X'X$. It will be seen that any point in the plane can be located by means of its coördinates, and that there will always be a point which will correspond to any pair of values we may choose, and that there will be only one such point.

56. In *locating* or *plotting* a point whose coördinates are given, some convenient unit of measure must first be chosen for x and for y. These need not be the same, but we shall take them the same, unless it is more convenient to do otherwise. Then measure off the proper number of these units from the origin along each axis in the direction indicated by the sign of the coördinate. Through the points thus determined draw lines parallel to the axes, and their intersection will locate the point whose coördinates were given.

Thus, the locations of the points, $A=(3, 4)$, $B=(-2, 6)$, $C=(-5, -7)$, $D=(6, -3)$, are shown in Fig. 3.

If either of the coördinates is 0, the point will lie on one of the axes.

Thus, the location of $E=(0, -5,)$ and of $F=(2, 0)$ is shown.

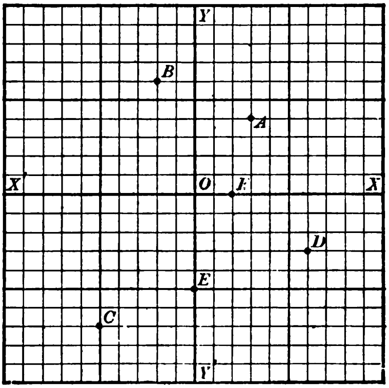


FIG. 3.

EXERCISE LVII

Locate the following points whose coördinates are:

1. $(2, 4)$.
2. $(-3, 4)$.
3. $(-3, -4)$.
4. $(-4, -8)$.
5. $(0, -9)$.
6. $(0, 0)$.
7. $(-4, -3\frac{1}{2})$.
8. $(-3, 2\frac{1}{2})$.
9. $(-7\frac{1}{2}; 9)$.
10. $(-10, 0)$.
11. $(5, 0)$.
12. $(5, 5)$.
13. $(-5, 5)$.
14. $(5, -5)$.
15. $(-5, -5)$.

GRAPHS OF SIMPLE EQUATIONS IN ONE UNKNOWN QUANTITY

57. If a single equation in x and y is given, it is evident that the coördinates of points taken at random will not satisfy it, since, if a value is assigned to one of the coördinates, the other will be determined by such an equation. There are then only certain points whose coördinates satisfy the given equation, and it will be discovered that these points lie consecutively, and hence form a curve (or straight line). Such a curve, which contains all the points which satisfy a given equation, is called the **graph** of that equation.

58. In case the equation contains only one unknown quantity, as $x = 5$, the graph is very easily determined, since the equation says that every point which satisfies it must have its x-coördinate equal to 5, but places no restriction upon the y-coördinate. All such points lie in MN, Fig. 4, 5 units to the right of the axis $Y'Y$, and MN is, therefore, the graph of the equation, $x = 5$. Similarly, the graph of any simple equation in one unknown can be shown to be a line parallel to one of the axes.

EXERCISE LVIII

Construct the graphs of the equations:

1. $x + 5 = 6$.
2. $y + 4 = 9$.
3. $9 + 5x = 16 + 4x$.
4. $8y = 5 + 10y - 11$.
5. $5x - (3x - 7) = 17$.
6. $\frac{y}{6} + 1 = 2$.
7. $\frac{x}{4} + \frac{x}{5} = 9$.
8. $\frac{3x+5}{4} - \frac{x}{2} = \frac{13-4x}{2}$.

GRAPHS OF SIMPLE EQUATIONS IN TWO UNKNOWNS

59. As the simplest type of equations in two unknowns, consider those in which the known quantity is wanting. Any such equation may be put into the form, $y = ax$, where a can have any value — positive, negative, or fractional. All points whose coördinates satisfy this equation must have their y-coördinate a times their x-coördinate, and hence must lie on a straight line through the origin, as KL in Fig. 4. To determine the graph of any such equation, plot any one point which satisfies it, and draw a line of indefinite length through this point and the origin.

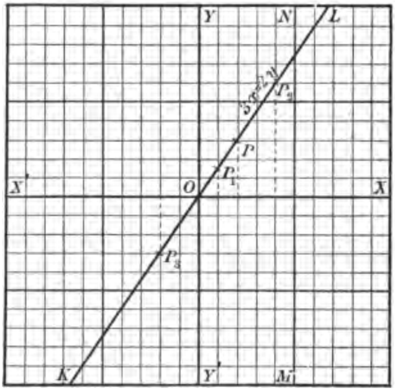


FIG. 4.

For example, the equation $3x = 2y$ is satisfied by (2, 3). Hence a line through $P = (2, 3)$ and (0, 0) is its graph. If the pupil has not had enough geometry to be sure that all points whose coördinates satisfy the equation must lie on the line, let him plot a number of such points, as $(1, \frac{3}{2})$, $(3, \frac{9}{2})$, (4, 6), etc., and convince himself that they all do lie on the line.

EXERCISE LIX

Construct the graphs of the following equations:

1. $y = 3x$.	3. $x + y = 0$.	5. $3x + 8y = 0$.
2. $5y = 7x$.	4. $2x - 5y = 0$.	6. $x - 5y = 0$.

60. Any simple equation in x and y, which contains a constant term, can be reduced to the form $y = ax + b$. If the graph of the equation $y = ax$ is plotted, and from every point on this line lines parallel to $Y'Y$ and equal in length to b are drawn, the extremities of these lines will evidently be the points whose coördinates satisfy the equation $y = ax + b$. These points are also on a straight line. It will be noticed that the graph of every equation of the first degree in x and y is a straight line. To find the graph, it is only necessary to determine two points and draw a line through them. These two points are usually taken on the axes.

For example, the equation $2x - 3y + 6 = 0$ is satisfied by $(-3, 0)$ and $(0, 2)$; its graph has the position of MN in Fig. 5. The pupil should assure himself by trial that this line contains every point which satisfies the given equation; for example, the points $(1, 2\frac{2}{3})$, $(2, 3\frac{1}{3})$, $(4, 4\frac{2}{3})$, $(-1, 1\frac{1}{3})$, etc.

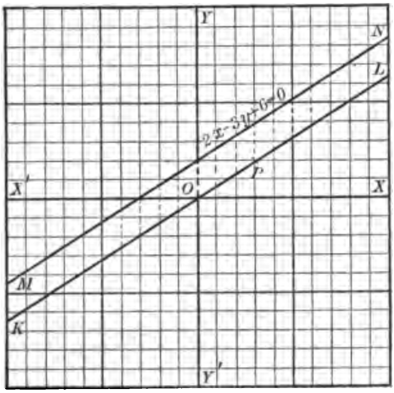


FIG. 5.

EXERCISE LX

Construct the graphs of the following equations:

1. $x + y = 3$.
2. $x + 5y = 16$.
3. $4x + y = 10$.
4. $3x + 2y = 13$.
5. $3x + 4y = 21$.
6. $4x + 5y = 25$.
7. $x + 6y = 20$.
8. $3x + 2y = 24$.

GRAPHS OF SIMULTANEOUS SIMPLE EQUATIONS

61. Consider the simultaneous simple equations:

$$\begin{cases} x+2y=4, & (1)\\ x+\ \ y=5. & (2)\end{cases}$$

In (1), $B=(0, 2)$, $A=(4, 0)$; in (2), $D=(0, 5)$, $C=(5, 0)$.

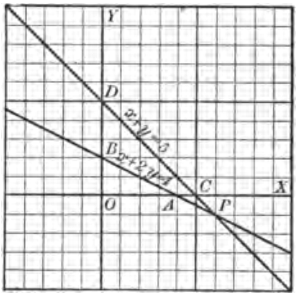


FIG. 6.

In Fig. 6, the location of the points A and B gives the line AB; and the location of the points C and D gives the line CD. The lines AB and CD intersect at P; and since P is on both lines, its coördinates must satisfy both equations. Hence its coördinates are the values of x and y which would be determined by solving the two equations simultaneously. These are found by measurement to be $x=6$ and $y=-1$. Two lines which intersect represent simultaneous equations which have a single solution.

62. Inconsistent equations may be shown to have no common solution by constructing their graphs.

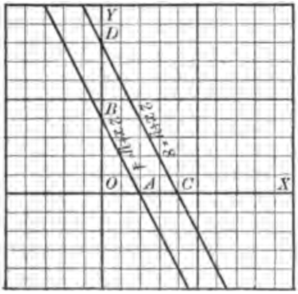


FIG. 7.

Thus, find a solution, if possible, of

$$\begin{cases} 2x+y=4, & (1)\\ 2x+y=8. & (2)\end{cases}$$

In (1), $B=(0, 4)$,
$A=(2, 0)$;
in (2), $D=(0, 8)$,
$C=(4, 0)$.

In Fig. 7 the graphs of (1) and (2) are such that they never meet; that is, AB and CD are parallel lines. Hence there is evidently no common solution of (1) and (2).

EXERCISE LXI

Determine the solutions of the following systems of equations by the graphical method, and test the accuracy of the result by solving the two equations algebraically.

1. $\begin{cases} x + y = 8, \\ x - 3y = 0. \end{cases}$

2. $\begin{cases} -x + y = 1, \\ 2x + y = 10. \end{cases}$

3. $\begin{cases} x + y = 14, \\ x - y = 2. \end{cases}$

4. $\begin{cases} 2x + 3y = 2, \\ -3x - 7y = 2. \end{cases}$

5. $\begin{cases} 4x + 3y = 10, \\ 2x + y = 6. \end{cases}$

6. $\begin{cases} 2x + 3y = 12, \\ 4x + 5y = 20. \end{cases}$

7. $\begin{cases} x + y = 5, \\ 2x + y = 6. \end{cases}$

8. $\begin{cases} 3x + 2y = 7, \\ 2x + 3y = 8. \end{cases}$

9. $\begin{cases} 2x - 3y = 2, \\ x - 5y = -5. \end{cases}$

10. $\begin{cases} 2x + 3y = 5, \\ 3x + 2y = 5. \end{cases}$

11. $\begin{cases} x - 2y = 4, \\ 2x - 4y = 8. \end{cases}$

12. $\begin{cases} 5x - 3y = -2, \\ 4x + 2y = -6. \end{cases}$

13. $\begin{cases} 3x + 2y = 6, \\ 9x + 6y = 18. \end{cases}$

14. $\begin{cases} 3x + 4y = 9, \\ 3x + 4y = 12. \end{cases}$

15. $\begin{cases} 4x + 2y = 8, \\ 10x + 5y = 20. \end{cases}$

16. $\begin{cases} 2x - 3y = 24, \\ 2x - 3y = 6. \end{cases}$

17. $\begin{cases} 2x + 5y = 10, \\ 4x - 3y = 12. \end{cases}$

18. $\begin{cases} 5x - 6y = 3, \\ 10x - 12y = 6. \end{cases}$

19. $\begin{cases} 2x - 3y = 0, \\ 3x - 4y = 0. \end{cases}$

20. $\begin{cases} 4x - 5y = 1, \\ 5x - 4y = 9. \end{cases}$

21. $\begin{cases} 5x + 3y = 5, \\ 9x + 4y = 9. \end{cases}$

22. $\begin{cases} 6x - 5y = -7, \\ -2x + y = 3. \end{cases}$

23. $\begin{cases} 3x + 4y = 2, \\ 4x + 3y = 6. \end{cases}$

24. $\begin{cases} 3x - 5y = 12, \\ 6x - 10y = 24. \end{cases}$

GRAPHS OF EQUATIONS OF THE SECOND DEGREE

63. By methods similar to those employed in the previous articles it is possible to construct the graph of any equation of the second degree in two unknowns.

Consider the equation $y = ax^2 + bx + c$, the right hand member of which is evidently a part of the general form of complete quadratic equation in one unknown. The graphs of certain numerical forms of $y = ax^2 + bx + c$ for various characters of the roots are interesting.

64. (1) When $b^2 > 4ac$, and when $\sqrt{b^2 - 4ac}$ is rational.

1. Plot the graph of $y = x^2 - 4x + 3$.

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
$x=$	-2	-1	0	1	2	3	4	5	6
$y=$	15	8	3	0	-1	0	3	8	15

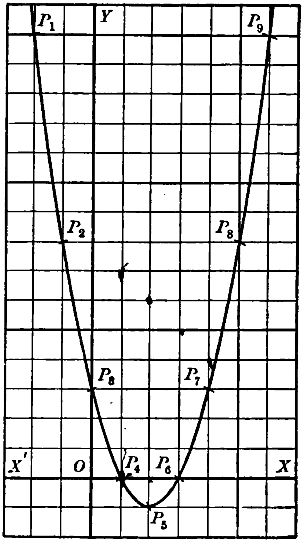


FIG. 8.

In the table are found the coördinates of the various points. Locating convenient points, and drawing a smooth curve through these points, the curve, $P_1P_2P_3P_4P_5P_6P_7P_8P_9$, Fig. 8, is the graph. The graph is seen to cut the X-axis at the points P_4 and P_6, whose coördinates are respectively (1, 0) and (3, 0). But, since the y-coördinates of the points where the graph crosses the X-axis are zero, the x-coördinates of these points are the solutions of the equation, $x^2 - 4x + 3 = 0$.

In like manner, if the graph of any equation in the form $y = ax^2 + bx + c$ is plotted, the x-coördinates of the points where the graph crosses the X-axis, will evidently be the solutions of $ax^2 + bx + c = 0$. The nature and approximate values of the solutions can therefore be determined from the graph.

65. (2) When $b^2 > 4\,ac$, and when $\sqrt{b^2 - 4\,ac}$ is irrational. Plot the graph of $y = x^2 - 4\,x + 2$.

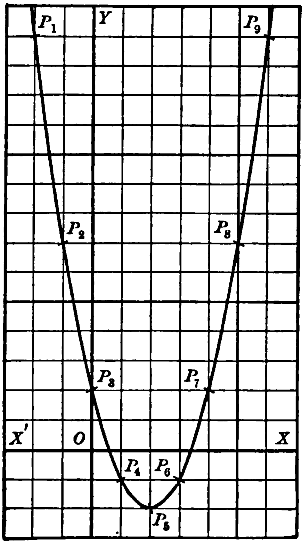


FIG. 9.

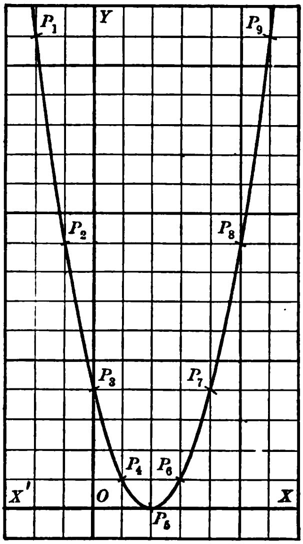


FIG. 10.

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
$x=$	-2	-1	0	1	2	3	4	5	6
$y=$	14	7	2	-1	-2	-1	2	7	14

The graph is constructed as shown in Fig. 9, and is seen to cut the X-axis at points whose x-coördinates are between 0 and 1, and between 3 and 4. By the usual method of solving the equation $x^2 - 4\,x + 2 = 0$, the roots are found to be $2 \pm \sqrt{2}$, or

0.586^{-} and 3.414^{+}. These must therefore be the exact values of x where the graph crosses the X-axis.

66. (3) When $b^2 = 4ac$.

Plot the graph of $y = x^2 - 4x + 4$.

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
$x=$	-2	-1	0	1	2	3	4	5	6
$y=$	16	9	4	1	0	1	4	9	16

Here the equation,

$$x^2 - 4x + 4 = 0,$$

has equal roots, $x = 2$, and the graph, Fig. 10, touches the X-axis at the single point P_5, whose coördinates are (2, 0).

67. (4) When $b^2 < 4ac$.

Plot the graph of

$$y = x^2 - 4x + 5.$$

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
$x=$	-2	-1	0	1	2	3	4	5	6
$y=$	17	10	5	2	1	2	5	10	17

If $y = 0$, and the resulting equation, $x^2 - 4x + 5 = 0$, is solved, the roots are found to be $x = 2 \pm \sqrt{-1}$. Since these values are imaginary, they cannot represent any real distance. Hence the graph, Fig. 11, does not cut the X-axis.

The graphs of the equations which have been plotted have the same general shape, which will be found to be the same for all equations of the form $y = ax^2 + bx + c$. This curve is called the **parabola**.

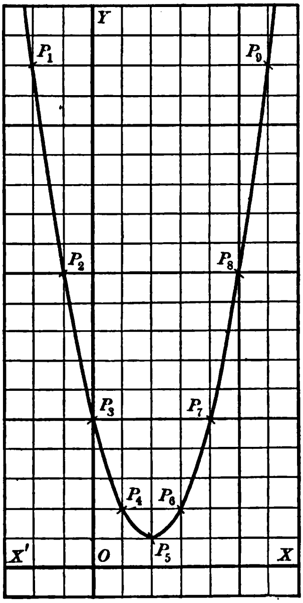


FIG. 11.

GRAPHS OF EQUATIONS CONTAINING y^2

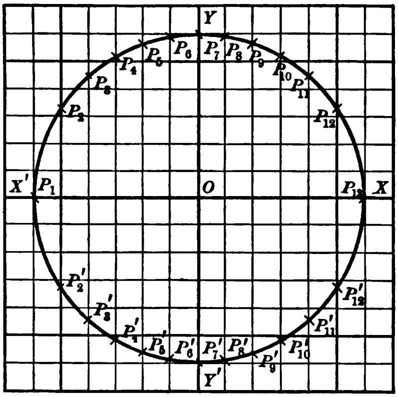


Fig. 12.

68. 1. Plot the graph of $x^2+y^2=36$.

Solving $x^2+y^2=36$, $y=\pm\sqrt{36-x^2}$. The nature of $\sqrt{36-x^2}$ is such that if x takes any values less than -6 or greater than 6, y becomes imaginary. It is necessary to construct a table only for values of x between -6 and $+6$.

$x=$	± 6	± 5	± 4	± 3	± 2	± 1	0
$y=$	0	$\pm\sqrt{11}$	$\pm 2\sqrt{5}$	$\pm 3\sqrt{3}$	$\pm 4\sqrt{2}$	$\pm\sqrt{35}$	6

The graph is constructed as shown in Fig. 12, using approximations of the double values of the surd values of y. Points may be located closer together by taking fractional values of x, as $1\frac{1}{2}$, $1\frac{3}{4}$, etc. The graph is seen to be a circle.

EXERCISE LXII

Plot the graphs of the following equations, and in 1–4 determine the approximate values of the roots of the quadratic equation formed by making y equal to zero.

1. $y=x^2-7x+10$.
2. $y=x^2-3x+5$.
3. $y=x^2-2x+1$.
4. $y=x^2-7x+4$.
5. $x^2+y^2=25$.
6. $x^2+y^2=16$.
7. $y^2=4x$.
8. $x^2+y^2-x-8=0$.

CHAPTER VIII

SIMULTANEOUS EQUATIONS SOLVABLE BY QUADRATICS TWO UNKNOWN QUANTITIES

69. A system of two simultaneous quadratic equations involving two unknown quantities cannot, in general, be solved by quadratics.

Solve the equations:
$$\begin{cases} x^2 - y^2 = 3, & (1) \\ x^2 + 2x + y = 8. & (2) \end{cases}$$

Substituting in (2) the value of y in (1) and simplifying,

$$x^4 + 4x^3 - 13x^2 - 32x + 67 = 0. \qquad (3)$$

Equation (3) cannot be solved by the method of quadratics; and, in general, the solution of a pair of quadratic equations, chosen at random, will involve the solution of an equation of the fourth degree.

There are, however, certain forms of simultaneous equations which can be solved by means of quadratics.

SIMULTANEOUS EQUATIONS SOLVABLE BY QUADRATICS

70. In § 61 it was shown that the coördinates of the point of intersection of two lines are the values of x and y in the solution of the two equations which the lines represent, since the coördinates of this point must satisfy both equations. For the same reason, if the graphs of two quadratic equations or a simple and a quadratic equation are plotted, the coördinates of the points of intersection of these graphs must be the solutions of the pair of equations.

71. 1. Plot the graphs of the system: $\begin{cases} x+y=3, & (1) \\ x^2+y=5. & (2) \end{cases}$

By § 60, construct the graph, AB, of $x+y=3$.
By § 68, construct the graph of $x^2+y=5$.

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
$x=$	0	± 1	$\pm\sqrt{2}$	$\pm\sqrt{3}$	± 2	$\pm\sqrt{5}$	$\pm\sqrt{6}$	$\pm\sqrt{7}$
$y=$	5	4	3	2	1	0	-1	-2

Locate suitable points and draw the smooth curve $P_8P_1P'_8$. The intersections of the graphs AB and the smooth curve $P'_8P_1P_8$, Fig. 13, will be points whose coördinates are solutions of the given system.

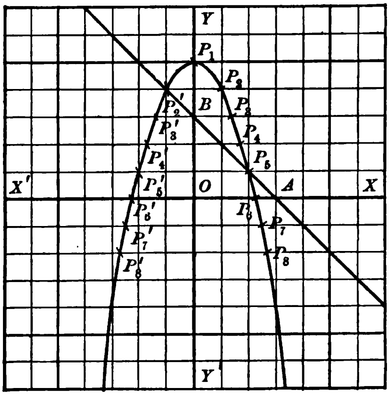


Fig. 13.

If, in place of $x+y=3$, the graph of $x+y=6$ is plotted, the graph will be found not to cut the parabola which is the graph of $x^2+y=5$. Corresponding to this non-intersection of the two graphs are found imaginary values for x and y when the equations $\begin{Bmatrix} x^2+y=5 \\ x+y=6 \end{Bmatrix}$ are solved simultaneously.

If the graph of $2x+y=6$ is plotted, the graph will be found just to touch the parabola at the point (1, 4). Corresponding to this fact, if the equations $\begin{cases} x^2+y=5 \\ 2x+y=6 \end{cases}$ are solved simultaneously, they have the single solution, $x=1$, $y=4$.

72. 1. Plot the graphs of the system: $\begin{cases} x^2+2y^2=64, & (1) \\ x^2-y^2=16. & (2) \end{cases}$

By the same method used in the preceding paragraph the graphs of the two equations are plotted as shown in Fig. 14.

They intersect in the four points, P_1, P_2, P_3, P_4, whose coördinates are found by measurement to agree with the solutions of the two equations,

$(4\sqrt{2}, 4)$, $(4\sqrt{2}, -4)$, $(-4\sqrt{2}, 4)$, $(-4\sqrt{2}, -4)$.

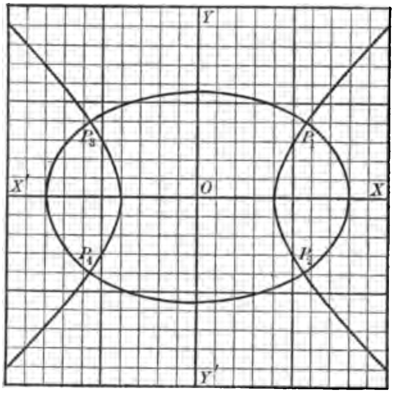


Fig. 14.

The graph of (1) is called an **ellipse**; of (2), an **hyperbola.**

EXERCISE LXIII

Plot the graphs of the following systems and determine by measurement the coördinates of their points of intersection.

1. $\begin{cases} x+y=2, \\ xy=1. \end{cases}$

2. $\begin{cases} x+y=4, \\ x^2+y^2=10. \end{cases}$

3. $\begin{cases} x^2+y^2=5, \\ xy=2. \end{cases}$

4. $\begin{cases} x^2-y^2=16, \\ x+y=8. \end{cases}$

5. $\begin{cases} x^2-y^2=5, \\ xy=6. \end{cases}$

6. $\begin{cases} x^2-y^2=24, \\ 3x^2-20y^2=55. \end{cases}$

7. $\begin{cases} 4x^2-xy=6, \\ 3xy-y^2=5. \end{cases}$

8. $\begin{cases} x^2+y^2=17, \\ x^2-y^2=15. \end{cases}$

CASE I

73. A simple equation and a quadratic equation.

A system of simultaneous equations in which one equation is simple and the other quadratic can always be solved by substituting in the quadratic equation the value of one of the unknowns obtained from the simple equation.

1. Solve the equations : $\begin{cases} x + y = 7, & (1) \\ x^2 + 2y^2 = 34. & (2) \end{cases}$

Substituting in (2), $x = 7 - y$ from (1),

$$(7 - y)^2 + 2y^2 = 34, \qquad (3)$$

simplifying in (3), $3y^2 - 14y + 15 = 0,$ (4)

factoring in (4), $(y - 3)(3y - 5) = 0,$ (5)

from (5), $y = 3$, or $y = \frac{5}{3}$. (6)

Substituting values of y from (6) in (1),

$$x = 4, \text{ or } x = \frac{16}{3}. \qquad (7)$$

The given equations check for $x = 4$ and $y = 3$, also for $x = \frac{16}{3}$ and $y = \frac{5}{3}$. A set of values of the unknowns which satisfies the given equations is called a solution of those equations.

*The values of the second unknown should always be found by substituting the value of the unknown first found in the **simple** equation, and **never in the quadratic equation.***

Let the student plot the graph of the above equations and test the accuracy of the solutions by measurement.

74. The double signs, ±, read "plus or minus," and ∓, read "minus or plus," taken together are to be interpreted in the order in which the signs are read.

Thus, $\begin{cases} x = \pm 1, \\ y = \pm 2, \end{cases}$ is equivalent to $\begin{cases} x = +1, \\ y = +2, \end{cases}$ and $\begin{cases} x = -1, \\ y = -2. \end{cases}$

Similarly $\begin{cases} x = \mp 1, \\ y = \pm 2, \end{cases}$ is equivalent to $\begin{cases} x = -1, \\ y = +2, \end{cases}$ and $\begin{cases} x = +1, \\ y = -2. \end{cases}$

EXERCISE LXIV

Solve the following systems of equations :

1. $\begin{cases} x - y = 3, \\ 3x^2 - 11y^2 = 1. \end{cases}$

2. $\begin{cases} xy - 5x = 1, \\ 7x - y = 1. \end{cases}$

3. $\begin{cases} x + 2y = 3, \\ 2x^2 + xy = 3. \end{cases}$

4. $\begin{cases} x - y = 4, \\ \dfrac{3x - 2}{y + 5} + \dfrac{y}{x} = 2. \end{cases}$

5. $\begin{cases} 2x - 3y = 2, \\ 3x^2 - 2y^2 = 115. \end{cases}$

6. $\begin{cases} x^2 + y^2 = 50, \\ 9x + 7y = 80. \end{cases}$

7. $\begin{cases} x + 2y = 3, \\ 2x^2 + y^2 = \frac{18}{35}. \end{cases}$

8. $\begin{cases} x - \dfrac{x - y}{2} = 4, \\ y - \dfrac{x + 3y}{x + 2} = 1. \end{cases}$

9. $\begin{cases} 4x + 3y = 1, \\ \dfrac{5y}{x} + \dfrac{2y + 3}{x + y} = 0. \end{cases}$

10. $\begin{cases} \dfrac{1}{x(b - a)} - \dfrac{3}{y(a + b)} + \dfrac{1}{a^2 - b^2} = 0, \\ \dfrac{a}{y + 4b} = \dfrac{2b}{x - y}. \end{cases}$

CASE II

75. When one of two simultaneous quadratic equations is homogeneous.

A quadratic equation is said to be **homogeneous** when all of the terms involved are of the second degree in the unknown quantities.

Thus, $x^2 - 3xy + 2y^2 = 0$ is a homogeneous quadratic equation.

1. Solve the system : $\begin{cases} x^2 - xy - 2y^2 = 0, & (1) \\ x^2 + y = 5. & (2) \end{cases}$

Dividing (1) by y^2, $$\left(\frac{x}{y}\right)^2 - \left(\frac{x}{y}\right) - 2 = 0, \qquad (3)$$

factoring (3), $$\left(\frac{x}{y} - 2\right)\left(\frac{x}{y} + 1\right) = 0, \qquad (4)$$

from (4), $x = 2y$, or $x = -y$, (5)

substituting $x = 2y$ in (2), $4y^2 + y = 5$, (6)

solving (6), $y = 1$, or $y = -\frac{5}{4}$, (7)

substituting values of y from (7) in (5), $x = 2$, or $x = -\frac{5}{2}$, (8)

substituting $x = -y$ in (2), $y^2 + y = 5$, (9)

solving (9), $y = \dfrac{-1 \pm \sqrt{21}}{2}$, (10)

substituting $y = \dfrac{-1 \pm \sqrt{21}}{2}$ in (5), $x = \dfrac{1 \mp \sqrt{21}}{2}$. (11)

The solutions are:

$$\begin{cases} x = 2, \\ y = 1, \end{cases} \quad \begin{cases} x = -\frac{5}{2}, \\ y = -\frac{5}{4}, \end{cases} \quad \begin{cases} x = \dfrac{1 - \sqrt{21}}{2}, \\ y = \dfrac{-1 + \sqrt{21}}{2}, \end{cases} \quad \begin{cases} x = \dfrac{1 + \sqrt{21}}{2}, \\ y = \dfrac{-1 - \sqrt{21}}{2}. \end{cases}$$

EXERCISE LV

Solve the following systems of equations:

1. $\begin{cases} x^2 + xy - 6y^2 = 0, \\ 2x^2 + 3y^2 = 11. \end{cases}$

2. $\begin{cases} 2x^2 - 3xy + y^2 = 0, \\ y^2 - x + 2y = 6. \end{cases}$

3. $\begin{cases} x^2 + xy = 0, \\ 2x^2 - 3x - y = 4. \end{cases}$

4. $\begin{cases} 2x^2 + xy - 10y^2 = 0, \\ x^2 + 3xy + y = -7. \end{cases}$

5. $\begin{cases} x^2 - 3xy + 2y^2 = 0, \\ xy - x + y = 4. \end{cases}$

6. $\begin{cases} x^2 - 9y^2 = 0, \\ 3x^2 + x - y = 29. \end{cases}$

7. $\begin{cases} x^2 - xy = 20y^2, \\ x^2 - x + y = 54. \end{cases}$

8. $\begin{cases} x^2 + y^2 = 2xy, \\ 2x^2 - xy + y = 30. \end{cases}$

9. $\begin{cases} x^2 - xy - 2y^2 = 0, \\ 2x^2 - 3x + y = 3. \end{cases}$

10. $\begin{cases} 5x^2 + 11xy + 2y^2 = 0, \\ x^2 - xy + y = 5. \end{cases}$

11. $\begin{cases} 3x^2 + 2xy - y^2 = 0, \\ x - 2y + 3y^2 = 32. \end{cases}$

12. $\begin{cases} 15x^2 - 34xy + 15y^2 = 0, \\ x + y - 2y^2 = -10. \end{cases}$

13. $\begin{cases} 8x^2 + 2xy - 3y^2 = 0, \\ x^2 + x + y^2 = 22. \end{cases}$

14. $\begin{cases} 3x^2 + 8xy + 5y^2 = 0, \\ 3x^2 + 4xy + y = -30. \end{cases}$

15. $\begin{cases} 2x^2 + 9xy = 35y^2, \\ 2x(x + y) - 11y = 236. \end{cases}$

16. $\begin{cases} 6x^2 = 11xy + 35y^2, \\ x^2 - 17xy - 180y = -260. \end{cases}$

17. $\begin{cases} 9x^2 - 39xy + 22y^2 = 0, \\ 3x^2 - 7x + y = 289. \end{cases}$

18. $\begin{cases} 10x^2 + 23xy + 12y^2 = 0, \\ 9x^2 + 7xy + 6y = 132. \end{cases}$

CASE III

76. When each of two simultaneous quadratic equations is homogeneous only in the unknowns involved.

A system of two simultaneous quadratic equations which are homogeneous except in the absolute terms may be solved as in Case II, by combining such multiples of the two equations as will make equal the absolute terms.

1. Solve the system : $\begin{cases} x^2 + xy = 12, & (1) \\ xy - 2y^2 = 1. & (2) \end{cases}$

Multiplying (2) by 12, $\quad 12xy - 24y^2 = 12, \quad (3)$

subtracting (3) from (1), $\quad x^2 - 11xy + 24y^2 = 0. \quad (4)$

Equation (4) may be solved as in Case II; or it may be solved by factoring.

Factoring (4), $\quad (x - 3y)(x - 8y) = 0, \quad (5)$

from (5), $\quad x = 3y$, or $x = 8y, \quad (6)$

substituting $x = 3y$ and $x = 8y$ in (1) and solving the resulting equations, $\quad y = \pm 1,\ y = \pm \frac{1}{6}\sqrt{6},$

by substitution in (6), $\quad x = \pm 3,\ x = \pm \frac{4}{3}\sqrt{6}.$

The solutions are : $\begin{cases} x = 3, \\ y = 1, \end{cases} \begin{cases} x = -3, \\ y = -1, \end{cases} \begin{cases} x = \frac{4}{3}\sqrt{6}, \\ y = \frac{1}{6}\sqrt{6}, \end{cases} \begin{cases} x = -\frac{4}{3}\sqrt{6}, \\ y = -\frac{1}{6}\sqrt{6}. \end{cases}$

77. An alternative method for solving equations of the class of Case III is called the ***vx* method.**

1. Solve the system : $\begin{cases} x^2 + xy + 4y^2 = 6, & (1) \\ 3x^2 + 8y^2 = 14. & (2) \end{cases}$

Let $y = vx$, and substitute in (1) and in (2),

$$x^2 + x^2v + 4x^2v^2 = 6, \quad (3) \qquad 3x^2 + 8x^2v^2 = 14, \quad (4)$$

factoring, $\quad x^2(1 + v + 4v^2) = 6, \quad (5) \qquad x^2(3 + 8v^2) = 14, \quad (6)$

$$x^2=\frac{6}{1+v+4v^2}, \quad (7) \qquad\qquad x^2=\frac{14}{3+8v^2}, \quad (8)$$

equating x^2 in (7) and (8), $$\frac{6}{1+v+4v^2}=\frac{14}{3+8v^2}, \tag{9}$$

clearing and simplifying in (9), $$4v^2+7v-2=0, \tag{10}$$

from (10), $$v=\tfrac{1}{4}, \text{ or } v=-2, \tag{11}$$

substituting values of v from (11) in (7),

$$\begin{cases} x^2=\dfrac{6}{1+\frac{1}{4}+\frac{1}{4}}=4, & (12) \\[2ex] x^2=\dfrac{6}{1-2+16}=\dfrac{6}{15}, & (13) \end{cases}$$

extracting square roots in (12) and in (13),

$$\begin{cases} x=\pm 2, & (14) \\ x=\pm \frac{1}{5}\sqrt{10}. & (15) \end{cases}$$

When $v=\frac{1}{4}$, $x=\pm 2$; substituting $v=\frac{1}{4}$ in $y=vx$,

$$y=\tfrac{1}{4}(\pm 2)=\pm\tfrac{1}{2}. \tag{16}$$

When $v=-2$, $x=\pm\frac{1}{5}\sqrt{10}$; substituting $v=-2$ in $y=vx$,

$$y=\mp\tfrac{2}{5}\sqrt{10}. \tag{17}$$

The solutions are:

$$\begin{cases} x=2, \\ y=\frac{1}{2}, \end{cases} \quad \begin{cases} x=-2, \\ y=-\frac{1}{2}, \end{cases} \quad \begin{cases} x=\frac{1}{5}\sqrt{10}, \\ y=-\frac{2}{5}\sqrt{10}, \end{cases} \quad \begin{cases} x=-\frac{1}{5}\sqrt{10}, \\ y=\frac{2}{5}\sqrt{10}. \end{cases}$$

The values of x must always be substituted in $y=vx$.

Since equations of the type of Case III may be reduced to a quadratic equation homogeneous in all its terms, and since such an equation may always be expressed as a quadratic in $\frac{x}{y}$, for $\frac{x}{y}$ any quantity v may be substituted. If $\frac{x}{y}=v$, $x=vy$; if $\frac{y}{x}=v$, $y=vx$.

EXERCISE LXVI

Solve the following systems of equations:

1. $\begin{cases} x^2 + 3xy = 27, \\ xy + 2y^2 = 14. \end{cases}$

2. $\begin{cases} x^2 + y^2 = 20, \\ x^2 - xy = 8. \end{cases}$

3. $\begin{cases} xy + 4 = 0, \\ 9x^2 - y^2 = 7. \end{cases}$

4. $\begin{cases} x^2 - xy = 15, \\ x^2 - y^2 = 21. \end{cases}$

5. $\begin{cases} 2x^2 + xy = 52, \\ 2y^2 - xy = 30. \end{cases}$

6. $\begin{cases} x^2 + xy = 77, \\ xy - y^2 = 12. \end{cases}$

7. $\begin{cases} xy = 4 - y^2, \\ 2x^2 - y^2 = 17. \end{cases}$

8. $\begin{cases} 2x^2 - 3y^2 = 60, \\ 3x^2 - 4xy + y^2 = 64. \end{cases}$

9. $\begin{cases} 6x^2 - 5xy + 2y^2 = 12, \\ 3x^2 + 2xy - 3y^2 = -3. \end{cases}$

10. $\begin{cases} 2x^2 - 2xy - y^2 = 3, \\ x^2 + 3xy + y^2 = 11. \end{cases}$

11. $\begin{cases} 3x^2 - 7xy + 4y^2 = -1, \\ 2x^2 + xy - 3y^2 = 22. \end{cases}$

12. $\begin{cases} 3x^2 - 3xy + y^2 = 21, \\ x^2 - 2xy + 3y^2 = 19. \end{cases}$

CASE IV

78. When two simultaneous quadratic equations are each symmetric with respect to the unknowns involved.

An equation is said to be **symmetric** with respect to the unknowns involved when the interchange of the unknowns does not change the form of that equation.

Thus, $x^2 + xy + y^2 = 7$, and $xy + x + y = 5$, are symmetric quadratic equations.

A solution of a system of such equations may always be found by substituting $x = u + v$, and $y = u - v$, in the given equations.

Solve the system: $\begin{cases} x^2 + xy + y^2 = 7, & (1) \\ xy + x + y = 5. & (2) \end{cases}$

Let $x = u + v$, and let $y = u - v$.

Substituting $x = u + v$, and $y = u - v$, in (1) and in (2),

$$(u+v)^2 + (u+v)(u-v) + (u-v)^2 = 7, \tag{3}$$

$$(u+v)(u-v) + (u+v) + (u-v) = 5, \tag{4}$$

simplifying in (3) and in (4), $$3u^2 + v^2 = 7, \tag{5}$$

$$u^2 + 2u - v^2 = 5, \tag{6}$$

transposing in (5) and in (6), $$v^2 = 7 - 3u^2, \tag{7}$$

$$v^2 = u^2 + 2u - 5, \tag{8}$$

equating v^2 in (7) and in (8), $$7 - 3u^2 = u^2 + 2u - 5, \tag{9}$$

solving (9), $$u = \tfrac{3}{2}, \text{ or } u = -2. \tag{10}$$

Substituting $u = \frac{3}{2}$, in (5), $v = \pm\frac{1}{2}$,

$$x = u + v = \tfrac{3}{2} \pm \tfrac{1}{2} = 2, \text{ or } 1,$$

and, $$y = u - v = \tfrac{3}{2} \mp \tfrac{1}{2} = 1, \text{ or } 2.$$

Substituting $u = -2$, in (5), $v = \pm\sqrt{-5}$,

$$x = u + v = -2 \pm \sqrt{-5},$$

and, $$y = u - v = -2 \mp \sqrt{-5}.$$

The solutions are:

$$\begin{cases} x = 2, \\ y = 1, \end{cases} \quad \begin{cases} x = 1, \\ y = 2, \end{cases} \quad \begin{cases} x = -2 + \sqrt{-5}, \\ y = -2 - \sqrt{-5}, \end{cases} \quad \begin{cases} x = -2 - \sqrt{-5}, \\ y = -2 + \sqrt{-5}. \end{cases}$$

Two simultaneous quadratic equations which are symmetric, except in respect to signs, can often be solved by Case IV.

EXERCISE LXVII

Solve the following systems of equations:

1. $\begin{cases} x + xy + y = 29, \\ x^2 + xy + y^2 = 61. \end{cases}$ **2.** $\begin{cases} 3(x^2 + y^2) - 5\,xy = 15, \\ 3(x + y) = 4\,xy. \end{cases}$

3. $\begin{cases} x^2 + y^2 - x - y = 22, \\ x + y + xy = -1. \end{cases}$

4. $\begin{cases} xy + x(x + 1) + y(y + 1) = 24, \\ xy = 6. \end{cases}$

5. $\begin{cases} xy - 2\,x^2 - 2\,y^2 = -20, \\ 4\,xy + x + y = 29. \end{cases}$

6. $\begin{cases} 3\,x^2 + 3\,y^2 = 8(x + y) - 1, \\ xy - x - y = 1. \end{cases}$

7. $\begin{cases} x^2 + y^2 + xy + x + y = 17, \\ x^2 + y^2 - 3\,xy + 2\,x + 2\,y = 9. \end{cases}$

8. $\begin{cases} 2\,x + 2\,y + xy = 16, \\ 3\,x(1 + x) + 3\,y(1 + y) = 54. \end{cases}$

9. $\begin{cases} x^2 + y^2 + x + y = 62, \\ 5\,xy + 4(x^2 + y^2) = 328. \end{cases}$

10. $\begin{cases} x^2 + 2\,xy + y^2 + 5\,x + 5\,y = 84, \\ x^2 + y^2 + x + y = 32. \end{cases}$

11. $\begin{cases} \dfrac{1}{x^2} + \dfrac{1}{xy} + \dfrac{1}{y^2} = 7, \\ x(x - y) + y\,(x + y) = \frac{3}{4} + xy. \end{cases}$

12. $\begin{cases} x^2 + y^2 + x + y = a^2, \\ xy + x + y = \dfrac{3\,a}{2}. \end{cases}$

SPECIAL DEVICES

79. Special devices may be employed in finding solutions by shorter methods for some of the systems in the preceding cases, as well as for certain other systems whose equations are often of higher degree than the second.

1. Solve the system:
$$\begin{cases} x + y = 3, & (1) \\ x^2 + y^2 = 29. & (2) \end{cases}$$

Squaring (1) and subtracting from (2),
$$-2xy = 20, \qquad (3)$$
adding (3) and (2), $x^2 - 2xy + y^2 = 49$, (4)

extracting square roots in (4), $x - y = \pm 7$, (5)

adding (5) and (1), $x = 5$, or -2, (6)

subtracting (1) from (5), $y = -2$, or 5. (7)

The solutions are: $\begin{cases} x = 5, \\ y = -2, \end{cases}$ $\begin{cases} x = -2, \\ y = 5. \end{cases}$

2. Solve the system:
$$\begin{cases} x^3 + y^3 = 1001, & (1) \\ x + y = 11. & (2) \end{cases}$$

Dividing (1) by (2), $x^2 - xy + y^2 = 91$, (3)

squaring (2) and subtracting from (3),
$$-3xy = -30, \qquad (4)$$
dividing (4) by -3 and subtracting from (3),
$$x^2 - 2xy + y^2 = 81, \qquad (5)$$
extracting square roots in (5), $x - y = \pm 9$, (6)

combining (2) and (6), $x = 10$ or 1, $y = 1$ or 10. (7)

The solutions are: $\begin{cases} x = 10, \\ y = 1, \end{cases}$ $\begin{cases} x = 1, \\ y = 10. \end{cases}$

EXERCISE LXVIII

Solve the following systems of equations:

1. $\begin{cases} x + y = 6, \\ xy = 5. \end{cases}$

2. $\begin{cases} x + y = 20, \\ xy = 51. \end{cases}$

3. $\begin{cases} x^2 + y^2 = 170, \\ xy = 13. \end{cases}$

4. $\begin{cases} x^2 + y^2 = 34, \\ xy = 15. \end{cases}$

5. $\begin{cases} x^2 + y^2 = 25, \\ 2\,xy = 24. \end{cases}$

6. $\begin{cases} x + y = 12, \\ x^2 + y^2 = 74. \end{cases}$

7. $\begin{cases} x - y = 2, \\ x^2 - y^2 = 20. \end{cases}$

8. $\begin{cases} x^2 + y^2 = 34, \\ x + y = 8. \end{cases}$

9. $\begin{cases} x^2 + y^2 = 74, \\ x - y = 2. \end{cases}$

10. $\begin{cases} xy = a, \\ x^2 + y^2 = b. \end{cases}$

11. $\begin{cases} x + y = a, \\ x^2 - y^2 = b. \end{cases}$

12. $\begin{cases} y^2 + xy = 15, \\ x^2 + xy = 10. \end{cases}$

13. $\begin{cases} x^2 + y^2 = 436, \\ x - y = 14. \end{cases}$

14. $\begin{cases} x^2 + xy = 15, \\ xy + y^2 = 3. \end{cases}$

15. $\begin{cases} x^2 - xy + y^2 = 13, \\ x + y = -2. \end{cases}$

16. $\begin{cases} x^2 + 3\,xy = 28, \\ xy + 4\,y^2 = 8. \end{cases}$

17. $\begin{cases} x + xy + y = 29, \\ x^2 + xy + y^2 = 61. \end{cases}$

18. $\begin{cases} x^2 + xy + y^2 = 19, \\ x^2 - xy + y^2 = 7. \end{cases}$

19. $\begin{cases} 2\,x^2 + 5\,xy = 38, \\ 2\,y^2 - xy = 12. \end{cases}$

20. $\begin{cases} x^2 + 5\,xy + y^2 = 43, \\ x^2 + 5\,xy - y^2 = 25. \end{cases}$

21. $\begin{cases} x^3 - y^3 = 98, \\ x - y = 2. \end{cases}$

22. $\begin{cases} x^3 + y^3 = 35, \\ x + y = 5. \end{cases}$

23. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 7, \\[2ex] \dfrac{1}{x^2} + \dfrac{1}{y^2} = 25. \end{cases}$

24. $$\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=a, \\ \dfrac{1}{x^2}+\dfrac{1}{y^2}=b. \end{cases}$$

25. $$\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=5, \\ \dfrac{1}{x^2}+\dfrac{1}{y^2}=13. \end{cases}$$

26. $$\begin{cases} x^2-xy=153, \\ x+y=1. \end{cases}$$

27. $$\begin{cases} x+y=3, \\ x^4+y^4=17. \end{cases}$$

28. $$\begin{cases} x^2+xy=10, \\ xy-y^2=-3. \end{cases}$$

29. $$\begin{cases} x+y=-3, \\ \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{6}. \end{cases}$$

30. $$\begin{cases} 5\,xy=84-x^2y^2, \\ x-y=6. \end{cases}$$

31. $$\begin{cases} x+y+3=0, \\ x^2+2\,y^2=8. \end{cases}$$

32. $$\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=2, \\ xy+\dfrac{1}{x}+\dfrac{1}{y}=8. \end{cases}$$

33. $$\begin{cases} \dfrac{x^2+y^2}{x^2-y^2}=\dfrac{29}{21}, \\ x+y=7. \end{cases}$$

34. $$\begin{cases} x+\dfrac{1}{y}=3, \\ y+\dfrac{1}{x}=\dfrac{12}{5}. \end{cases}$$

35. $$\begin{cases} x+\dfrac{1}{y}=1, \\ y+\dfrac{1}{x}=4. \end{cases}$$

36. $$\begin{cases} xy-\dfrac{x}{y}=2, \\ xy-\dfrac{y}{x}=\dfrac{1}{2}. \end{cases}$$

37. $$\begin{cases} ay^2+bxy=b, \\ bx^2+axy=a. \end{cases}$$

38. $$\begin{cases} \dfrac{a^2}{x^2}+\dfrac{b^2}{y^2}=10, \\ \dfrac{ab}{xy}=3. \end{cases}$$

39. $$\begin{cases} \dfrac{x}{a}+\dfrac{y}{b}=1, \\ \dfrac{a}{x}+\dfrac{b}{y}=4. \end{cases}$$

40. $$\begin{cases} \sqrt{x+y}+\sqrt{x-y}=4, \\ x^2-y^2=9. \end{cases}$$

41. $$\begin{cases} x+y+\sqrt{x+y}=12, \\ xy=20. \end{cases}$$

42. $$\begin{cases} x^2y-xy^2=12, \\ x^3-y^3=63. \end{cases}$$

THREE OR MORE UNKNOWN QUANTITIES

80. Three simultaneous quadratic equations involving three unknown quantities cannot in general be solved by quadratic equations. The solutions of certain forms are illustrated in the following examples.

1. Solve the system:
$$\begin{cases} x^2+2y^2-z^2=5, & (1)\\ 2x+y+z=6, & (2)\\ x+4y-z=5. & (3)\end{cases}$$

Eliminating z in (2) and (3), $x=\dfrac{11-5y}{3}$, (4)

eliminating x in (2) and (3), $z=\dfrac{7y-4}{3}$, (5)

substituting x and z from (4) and (5) in (1),
$$\left(\frac{11-5y}{3}\right)^2+2y^2-\left(\frac{7y-4}{3}\right)^2=5, \qquad (6)$$

simplifying and solving (6), $y=1$, or $y=-10$. (7)

Substituting values of y in (4) and (5), $\begin{cases} x=2,\\ y=1,\\ z=1,\end{cases}$ $\begin{cases} x=\frac{61}{3},\\ y=-10,\\ z=-\frac{74}{3}.\end{cases}$

2. *Solve the system:*
$$\begin{cases} x(y+z)=-4, & (1)\\ y(x+z)=-10, & (2)\\ z(x+y)=-54. & (3)\end{cases}$$

Dividing the sum of (1), (2), and (3) by 2,
$$xy+yz+xz=-34, \qquad (4)$$

Subtracting (1) from (4), $y=-\dfrac{30}{z}$, (5)

Subtracting (2) from (4), $x=-\dfrac{24}{z}$, (6) substituting y and x from (5) and (6) in (4),

$$\frac{720}{z^2}-30-24=-34, \tag{7}$$

solving (7), $z=\pm 6$, (8)

substituting z from (8) in (6), $x=\mp 4$, (9)

substituting z from (8) in (5), $y=\mp 5$. (10)

EXERCISE LXIX

Solve the following systems of equations:

1. $\begin{cases} xy=-42, \\ xz=48, \\ yz=-56. \end{cases}$

2. $\begin{cases} x(y+z)=8, \\ y(z+x)=18, \\ z(x+y)=20. \end{cases}$

3. $\begin{cases} x^2+y^2=13, \\ y^2+z^2=25, \\ z^2+x^2=20. \end{cases}$

4. $\begin{cases} xy+xz+yz=3, \\ x+2y+3z=6, \\ 3x+2y+z=6. \end{cases}$

5. $\begin{cases} x+y=15, \\ u+z=3, \\ x+u^2=8, \\ y+z^2=12. \end{cases}$

6. $\begin{cases} x^2+xy+y^2=19, \\ y^2+yz+z^2=37, \\ z^2+xz+x^2=28. \end{cases}$

7. $\begin{cases} x^2+xy+z=2, \\ x+2y+z=3, \\ x-y+z=0. \end{cases}$

8. $\begin{cases} x^2+y^2+z^2=21, \\ xy+xz+yz=14, \\ x+y-z=-1. \end{cases}$

9. $\begin{cases} x+y+z=4, \\ xy+xz+yz=-4, \\ x-y+z=8. \end{cases}$

10. $\begin{cases} xy+zu=14, \\ xz+yu=11, \\ xu+yz=10, \\ x+y+z+u=10. \end{cases}$

CHAPTER IX

RATIO, PROPORTION, VARIATION

81. The **ratio of one number to another number** is the quotient obtained by dividing the first by the second number. The quotient shows how the numbers compare.

Thus, the ratio of 5 to 7 is indicated: $5 \div 7$, $\frac{5}{7}$, $5:7$.

The **ratio of one quantity to another quantity of the same kind** is the ratio of the numerical values of the quantities.

Thus, the ratio of a dollars to b dollars is $\frac{a}{b}$.

The **terms of a ratio** are the terms of the fraction indicating the ratio; the numerator is called the **antecedent,** and the denominator the **consequent** of the ratio.

Thus, a and b are the terms, a is the antecedent, and b the consequent of the ratio $\frac{a}{b}$.

There is *no ratio of one quantity to another of a different kind*, since it is impossible to compare such quantities.

Thus, no ratio exists between a inches and b pounds.

82. A **proportion** is an equation whose members are ratios. A proportion may be expressed thus: $\frac{a}{b}=\frac{c}{d}$, $a:b=c:d$, $a:b::c:d$.

The terms of the equal ratios forming a proportion are called the **terms of the proportion.** The antecedents and consequents of the ratios are called the **antecedents** and **consequents** of the proportion. The first and fourth terms of a proportion are called the **extremes** and the second and third terms are called the **means.** The terms of a proportion are said to be **proportional.** The fourth term of a proportion is called a **fourth proportional.** When the second and third terms of a proportion are identical, it is called a **mean proportional,** and the consequent of the second ratio is called a **third proportional.**

Thus, in the proportion, $\frac{a}{b}=\frac{c}{x}$, a, b, c, and x are proportional, x is a fourth proportional; in the proportion, $\frac{a}{b}=\frac{b}{x}$, x is a third proportional, and b is a mean proportional.

A **continued proportion** is a series of equal ratios in which the consequent of each ratio is the antecedent of the next ratio.

Thus, $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$ is a continued proportion.

PRINCIPLES OF PROPORTION

83. I. *In any proportion the product of the means equals the product of the extremes.*

If $$\frac{a}{b}=\frac{c}{d}, \qquad (1)$$

multiplying (1) by bd, $$ad = bc. \qquad (2)$$

II. *If two products are each composed of two factors, these factors form a proportion in which the factors of either product can be made the means, and the other two factors the extremes.*

Let the student prove this theorem and each of the following theorems to which no proof is given.

III. *The products of corresponding terms of two or more proportions are in proportion.*

IV. *The quotients of the corresponding terms of two proportions are in proportion.*

V. *If four quantities, a, b, c, d, are in proportion, they are in proportion by* ***inversion****; that is,* $\frac{b}{a}=\frac{d}{c}$.

VI. *If four quantities of the same kind, a b, c, d, are in proportion, they are in proportion by* ***alternation****; that is,*

$$\frac{a}{c}=\frac{b}{d}.$$

VII. *If four quantities, a, b, c, d, are in proportion, they are in proportion by* ***composition****; that is,* $\frac{a+b}{b}=\frac{c+d}{d}$, *or* $\frac{a+b}{a}=\frac{c+d}{c}$.

If $$\frac{a}{b}=\frac{c}{d}, \tag{1}$$

adding 1 to each member of (1),

$$\frac{a}{b}+1=\frac{c}{d}+1, \tag{2}$$

or rewriting (2), $$\frac{a+b}{b}=\frac{c+d}{d}. \tag{3}$$

Similarly, (1), written first by V, and then by composition, is $\frac{a+b}{a}=\frac{c+d}{c}$.

VIII. *If four quantities, a, b, c, d, are in proportion, they are in proportion by* ***division****; that is,* $\frac{a-b}{b}=\frac{c-d}{d}$, *or* $\frac{a-b}{a}=\frac{c-d}{c}$.

IX. *If four quantities, a, b, c, d, are in proportion, they are in proportion by* ***composition and division;*** *that is,* $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

If $$\frac{a}{b}=\frac{c}{d}, \qquad (1)$$

writing (1) by VII, $$\frac{a+b}{b}=\frac{c+d}{d}, \qquad (2)$$

writing (1) by VIII, $$\frac{a-b}{b}=\frac{c-d}{d}, \qquad (3)$$

by IV, $$\frac{a+b}{a-b}=\frac{c+d}{c-d}. \qquad (4)$$

X. *Like powers, or like roots of four quantities, a, b, c, d, which are in proportion, are in proportion; or* $\frac{a^n}{b^n}=\frac{c^n}{d^n}$.

XI. *In a series of equal ratios the sum of the antecedents is to the sum of the consequents as any antecedent is to its own consequent.*

If $$\frac{a}{b}=\frac{c}{d}=\frac{m}{n}=\frac{x}{y}, \qquad (1)$$

let $$\frac{a}{b}=r,\ \frac{c}{d}=r,\ \frac{m}{n}=r,\ \frac{x}{y}=r, \qquad (2)$$

clearing of fractions in (2), $a=br$, $c=dr$, $m=nr$, $x=yr$, (3)

adding, $a+c+m+x=(b+d+n+y)r$, (4)

dividing each member of (4) by $(b+d+n+y)$,

$$\frac{a+c+m+x}{b+d+n+y}=r=\frac{a}{b}=\frac{c}{d}=\frac{m}{n}=\frac{x}{y}. \qquad (5)$$

EXAMPLES

1. Solve for x, $\dfrac{\sqrt{2x+3}-\sqrt{2x-5}}{\sqrt{2x+3}+\sqrt{2x-5}}=\dfrac{1}{2}.$ (1)

By IX, $\dfrac{2\sqrt{2x+3}}{-2\sqrt{2x-5}}=\dfrac{3}{-1},$ (2)

simplifying (2), $\sqrt{2x+3}=3\sqrt{2x-5},$ (3)

solving (3), $x=3.$ (4)

2. If $\dfrac{a}{b}=\dfrac{c}{d},$ prove that $\dfrac{pa^2-qc^2}{pb^2-qd^2}=\dfrac{a^2}{b^2}.$ (1)

By X, $\dfrac{a^2}{b^2}=\dfrac{c^2}{d^2},$ or $\dfrac{pa^2}{pb^2}=\dfrac{-qc^2}{-qd^2},$ (2)

by XI, $\dfrac{pa^2-qc^2}{pb^2-qd^2}=\dfrac{pa^2}{pb^2}=\dfrac{a^2}{b^2}.$ (3)

EXERCISE LXX

1. Find a fourth proportional to 462, 77, and 90.

2. Find a third proportional to 35 and 91.

3. Find a mean proportional to $2+\sqrt{3}$ and $2-\sqrt{3}$.

4. Solve for x: $\dfrac{4x^2-5x+11}{3x-7}=\dfrac{4x^2-5x-7}{3x-16}.$

5. Solve for x: $\dfrac{5x^2-4x+10}{3x^2-x+1}=\dfrac{5x^2-4x-2}{3x^2-x-5}.$

6. Solve for a: $\dfrac{\sqrt{a+5}+\sqrt{5-a}}{\sqrt{a+5}-\sqrt{5-a}}=2.$

7. Solve for x:

$$\frac{\sqrt{a-bx}+\sqrt{c-mx}}{\sqrt{a-bx}+\sqrt{nx-d}}=\frac{\sqrt{a-bx}-\sqrt{c-mx}}{\sqrt{a-bx}-\sqrt{nx-d}}.$$

If $\frac{a}{b}=\frac{c}{d}$, prove that:

8. $\frac{a^2}{c^2}=\frac{a^2-b^2}{c^2-d^2}$.

9. $\frac{a^2+c^2}{b^2+d^2}=\frac{ac}{bd}$.

10. $\frac{ab+cd}{ab-cd}=\frac{a^2+c^2}{a^2-c^2}$.

11. $\frac{a^2+ab}{c^2+cd}=\frac{b^2-2ab}{d^2-2cd}$.

12. $\frac{a+b}{c+d}=\frac{a-b}{c-d}=\frac{a}{c}=\frac{b}{d}$.

13. $\frac{a+b+c+d}{a+b-c-d}=\frac{a-b+c-d}{a-b-c+d}$.

14. $\frac{\sqrt[3]{a^3+c^3}}{\sqrt[3]{b^3+d^3}}=\frac{c}{d}$.

15. $\frac{a^2+ab+b^2}{a^2-ab+b^2}=\frac{c^2+cd+d^2}{c^2-cd+d^2}$.

If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove that:

16. $\frac{a+c+e}{b+d+f}=\frac{a}{b}$.

17. $\frac{ha+kc+le}{hb+kd+lf}=\frac{a}{b}$.

18. $\frac{a^3+c^3+e^3}{b^3+d^3+f^3}=\frac{ace}{bdf}$.

19. $\frac{ma^2+nc^2+pe^2}{mb^2+nd^2+pf^2}=\frac{ac}{bd}$.

20. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, prove that $\frac{a+b}{b+c}=\frac{b+c}{c+d}$.

21. If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=k$, prove that $\frac{\sqrt{px^2+qy^2+rz^2}}{\sqrt{pa^2+qb^2+rc^2}}=k$.

22. If $\frac{a}{b}=\frac{c}{d}$, if x is a third proportional to a and b, and if y is a third proportional to c and d, prove that $\frac{x}{y}=\frac{b}{d}$.

23. What is the ratio of the mean proportional between a and b, to the mean proportional between c and d?

24. Two numbers are as $3:4$, and if 7 be subtracted from each, the remainders are as $2:3$. Find the numbers.

25. What two numbers whose difference is d are to each other as $a:b$?

26. Two numbers x and y (the first being negative) are in the ratio 8 to -9: if 16 be subtracted from each one, the resulting numbers are in the ratio -9 to 8; find the numbers.

27. If $\frac{x}{a-b}=\frac{y}{b-c}=\frac{z}{c-a}$, prove that $x+y+z=0$.

28. If $\frac{x}{a^2-bc}=\frac{y}{b^2-ca}=\frac{z}{c^2-ab}=1$, prove that

$$(a+b+c)(x+y+z)=a^3+b^3+c^3-3\,abc.$$

29. Bronze is an alloy containing 80 per cent of copper, 4 of zinc, and 16 of tin. A fused mass of bronze and brass is found to contain 74 per cent of copper, 16 of zinc, and 10 of tin. Brass is an alloy of copper and zinc; find the ratio of these metals in its composition and find the ratio of brass to bronze in the given mass.

30. Two equal vessels are filled with spirits and water, the ratios of the amount of spirit to the amount of water being $a:1$ and $b:1$ respectively. The two are mixed; show that the ratio of the amount of spirit to the amount of water in the mixture is $a+b+2\,ab:a+b+2$.

VARIATION

84. A quantity whose value is dependent upon the value of another quantity is called a **function** of that quantity.

Thus, if $y = 2x^3$, y is called a function of x.

A function of x is indicated in any of the following ways: $F(x)$, $f(x)$, $\phi(x)$, etc.

When the value of a quantity is always the same in a particular investigation, the quantity is called a **constant.**

Thus, x is a constant whose value is 2, in $2x + 5 = 7 + x$.

When the value of a quantity changes in a particular investigation, the quantity is called a **variable.**

Thus, in the expression $x^3 + 1$, x is a variable, since it may take any value.

The theory of the dependence of a quantity upon another quantity is called **variation,** or **functionality.** Only the simplest forms of variation are discussed in this chapter.

The symbol $\propto$, called the **symbol of variation,** is used to indicate variation.

Thus, $x \propto y$ is read "x varies as y."

KINDS OF VARIATION

85. 1. If the ratio of two variables is constant, the variables are said to be in **direct variation.**

Thus, when m is a constant, if $\frac{x}{y} = m$, x varies directly as y; or $x \propto y$.

The height of a column of mercury in a thermometer is known to vary as the temperature. If H and H' represent the different heights of the mercury when the temperatures are respectively T and T', $H \propto T$; or, $H : H' = T : T'$.

2. If the ratio of a variable to the reciprocal of a second variable is constant, the variables are said to be in **inverse variation.**

Thus, when m is a constant, if $x : \frac{1}{y} = m$, x varies inversely as y, or $x \propto \frac{1}{y}$.

The volume of a gas is known to vary inversely as the pressure. If V and V' represent the volumes of a gas under the respective pressures P and P', $V \propto \frac{1}{P}$; or, $V : V' = \frac{1}{P} : \frac{1}{P'}$ which may be more conveniently written $V : V' = P' : P$.

3. If the ratio of a variable to the product of two other variables is a constant, the first variable is said to be in **joint variation** with the other two variables.

Thus, when m is a constant, if $x : yz = m$, x varies jointly as y and z; or, $x \propto yz$.

The distance travelled depends upon the rate and the time. If D and D' represent the distances travelled when the rates and times are respectively R and R', T and T', $D \propto RT$; or, $D : D' = RT : R'T'$.

4. If the ratio of a variable to a second variable multiplied by the reciprocal of a third variable is a constant, the first variable is said to be in **direct and inverse variation** with the second and third variables.

Thus, when m is a constant, if $x : \left(y \cdot \frac{1}{z}\right) = m$, x is in direct and inverse variation with y and z; or, $x \propto \frac{y}{z}$.

The base of a rectangle is known to vary as the area divided by the altitude. If B and B' represent the bases when the areas and altitudes are respectively S and S', A and A', $B \propto S \cdot \frac{1}{A}$; or, $B : B' = \frac{S}{A} : \frac{S'}{A'}$.

PRINCIPLES OF VARIATION

86. I. *If* $x \propto y$, *and* $y \propto z$, *then* $x \propto z$.

Using m and n as constants, let

$$\frac{x}{y} = m, \text{ or } x = ym, \tag{1}$$

and let $$\frac{y}{z} = n, \text{ or } y = zn, \tag{2}$$

multiplying (1) and (2), $$xy = yzmn, \tag{3}$$

dividing (3) by yz, $$\frac{x}{z} = mn. \tag{4}$$

In (4), since m and n are constants, mn is also a constant; hence $x \propto z$.

II. *If* $x \propto y$, *and* $x' \propto y'$, *then* $xx' \propto yy'$.

Using m and n as constants, let

$$\frac{x}{y} = m, \tag{1}$$

and let $$\frac{x'}{y'} = n, \tag{2}$$

multiplying (1) and (2), $$\frac{xx'}{yy'} = mn. \tag{3}$$

Hence $xx' \propto yy'$.

Similarly, if $x \propto y$, $x^n \propto y^n$.

III. *If* $x \infty y$, *then* $kx \propto ky$.

Let k be either a constant or a variable; and let m be a constant. Let

$$\frac{x}{y} = m, \qquad (1)$$

multiplying $\frac{x}{y}$ in (1) by $\frac{k}{k}$, $\frac{kx}{ky} = m$. (2)

Hence $kx \propto ky$.

IV. *If* $x \propto yz$, *then* $y \propto \frac{x}{z}$, *and* $z \propto \frac{x}{y}$.

Using m as a constant, let

$$m = \frac{x}{yz}, \text{ or } myz = x, \qquad (1)$$

dividing (1) by mz, $y = \frac{x}{mz}$, (2)

dividing (2) by $\frac{x}{z}$, $\frac{y}{\frac{x}{z}} = \frac{1}{m}$. (3)

Hence $y \propto \frac{x}{z}$. Similarly, $z \propto \frac{x}{y}$.

V. *If* $x \propto y$ *when* z *is constant, and if* $x \propto z$ *when* y *is constant, then* $x \propto yz$ *when both* y *and* z *are variables.*

Let x, y, z; x', y', z; x'', y', z', be three sets of corresponding values of x, y, and z.

If z is constant, $\frac{x}{x'} = \frac{y}{y'}$, (1)

if y is constant, $\frac{x'}{x''} = \frac{z}{z'}$, (2)

multiplying (1) and (2), $\frac{x}{x''} = \frac{yz}{y'z'}$, (3) or, rewriting (3), $\dfrac{x}{yz} = \dfrac{x''}{y'z'}$. (4)

Hence $x \propto yz$.

EXAMPLES

1. If $x \propto y$, and if $x = 3$ when $y = 2$, find x when $y = 6$.

Let $\dfrac{x}{y} = m$, (1)

substituting in (1) $x = 3$, $y = 2$, $\frac{3}{2} = m$, (2)

substituting in (1) $y = 6$, $\dfrac{x}{6} = m = \dfrac{3}{2}$, (3)

solving (3), $x = 9$. (4)

2. If y varies inversely as the square of x, and if $y = 8$ when $x = 3$, find x when $y = 2$.

Let $\dfrac{y}{\frac{1}{x^2}} = m$, $yx^2 = m$, (1)

substituting in (1) $y = 8$ and $x = 3$, $72 = m$, (2)

substituting in (1) $y = 2$ and $m = 72$, $2x^2 = 72$, (3)

solving (3), $x = \pm 6$. (4)

3. If $x \propto \dfrac{y}{z}$, and if $x = 4$ when $y = 6$ and $z = 3$, what is the value of x when $y = 6$ and $z = 9$?

Let $\dfrac{x}{\frac{y}{z}} = m$, $\dfrac{xz}{y} = m$, (1)

substituting in (1) $x = 4$, $y = 6$, $z = 3$, $m = 2$, (2)

substituting in (1) $y = 6$, $z = 9$, $m = 2$, $\dfrac{9x}{6} = 2$, (3)

solving (3), $x = \frac{4}{3}$. (4)

4. The volume of a sphere varies as the cube of the radius, and the volume of a sphere is $1437\frac{1}{3}$ when the radius is 7. Find the volume of a sphere whose radius is 14.

Let V represent the volume and R the radius of the sphere.

Then $$\frac{V}{R^3} = m, \text{ or } V = mR^3, \tag{1}$$

substituting in (1) $V = 1437\frac{1}{3}$, and $R = 7$, $m = \frac{4312}{3 \cdot 7^3}$, (2)

hence $$\text{volume} = \frac{4312}{3 \cdot 7^3} \cdot 14^3 = \frac{4312}{3} \cdot 8 = 11498\frac{2}{3}. \tag{3}$$

EXERCISE LXXI

1. If $x \propto y$, and if $x = 5$ when $y = 4$, find x when $y = 9$.

2. If $x \propto \frac{1}{y^2}$, and if $x = 4$ when $y = 3$, find y when $x = 2$.

3. If $x \propto yz$, and if $x = 2$ when $y = 3$ and $z = 4$, find x when $y = 2$ and $z = 6$.

4. If $x \propto \frac{y}{z}$, and if $x = 16$ when $y = 3$ and $z = 8$, find z when $x = 12$ and $y = 2$.

5. If $x \propto \frac{1}{y} + \frac{1}{z}$, and if $x = 4$ when $y = 3$ and $z = 5$, find y when $x = 3$ and $z = 2$.

6. If x varies directly as y and inversely as z, and is equal to 4 when $y = 2$ and $z = 3$, what is the value of x when $y = 35$ and $z = 15$?

7. If $y = u - v$, if u varies as x, and v as x^2, and if $y = 2$ when $x = 1$, and $y = 3$ when $x = 2$, find the value of y in terms of x.

8. If $a^2 - b^2$ varies as c^2, and if $c = 2$ when $a = 5$ and $b = 3$, find the equation between a, b, and c.

9. If $x \propto y$, and $z \propto y$, prove that $x - z \propto y$.

10. If $x \propto y$, prove that $x^2 + y^2 \propto xy$.

11. If $x + y \propto x - y$, prove that $x^2 + y^2 \propto xy$.

12. If $x \propto y$, and $x \propto z$, and $x \propto w$, when z and w, y and w, y and z, are constants, prove that $x \propto yzw$.

13. The area of a circle varies as the square of the radius; show that the area of a circle of 5 feet radius is equal to the sum of a circle of 3 feet radius and another of 4 feet radius.

14. Knowing that the volume, V, of a gas varies directly as the temperature, T, when $T = 273° +$ the number of degrees in temperature (in the Centigrade System); if the volume of a certain gas is 400 c.c. when the temperature is 27° C., find the volume of the gas at 127° C.

15. Find, under the law given in the preceding example, the volume of a gas at 0° C., if the volume is 250 c.c. at 18° C.

16. Knowing that the volume, V, of a gas varies inversely as the pressure, P, upon it; if the volume of a gas is 100 c.c. when the pressure is 76 cm., find the volume when the pressure is 38 cm.

17. Under the conditions given in the preceding problem, if the volume of a gas is 600 c.c. when the pressure is 60 cm., find the pressure if the volume is 150 c.c.

18. Knowing that the intensity of illumination, I, varies inversely as the square of the distance, D; if a candle throws a certain amount of light on a screen 2 feet distant, what will be its relative illuminating power at a distance of 7 feet?

19. Under the conditions given in the preceding problem, if a candle and a gas flame are 12 feet apart, and if the gas flame is equivalent to 4 candles, where must a screen be placed on a line joining the candle and gas flame so that the screen may be equally illumined by each of them?

20. Knowing that $\frac{V_1P_1}{T_1} = \frac{V_2P_2}{T_2}$ where V_1, V_2, etc., are as given in Problems 14 and 16; if a mass of air at 0° C. has a volume of 600 c.c. at a pressure of 76 cm., find the volume when the temperature is 91° C. and the pressure is 190 cm.

21. Under the conditions given in the preceding problem, if the volume of a certain mass of air at 27° C. and under a pressure of 225 cm. is 2000 c.c., find its volume at 127° C. under a pressure of 75 cm.

22. Knowing that the amount of bending, B, of a rod varies jointly as the load, L, and the cube of the length, L', and inversely and jointly as the width, W, and the cube of the thickness, T, that is, $B \propto \frac{LL'^3}{WT^3}$; if a rod 8 feet long, 4 inches wide, 1 inch thick, is bent 0.2 inch by a weight of 50 pounds, how much would a weight of 5 pounds bend a rod of like material, 24 feet long, 8 inches wide, and 2 inches thick?

23. Under the conditions given in the preceding problem, if a beam 16 feet long, 8 inches wide, 4 inches thick, is bent $\frac{1}{2}$ inch by a weight of 1000 pounds, how much would a beam 10 feet long, 6 inches wide, 8 inches thick, be bent by the same weight?

CHAPTER X

PROGRESSIONS

ARITHMETICAL PROGRESSION

87. A succession of terms, each of which is obtained from the preceding term by the addition of the same positive or negative quantity (the **common difference**), is called an **arithmetical progression.**

Thus, 2, 5, 8, 11, etc., and -1, -2, -3, etc., are arithmetical progressions.

The first term is usually represented by a, and the common difference by d; hence the progression is a, $a+d$, $a+2d$, $a+3d$, etc. The number of terms in a progression is represented by n; and the nth term by l.

Since each term is formed from the preceding term by the addition of d, the coefficient of d, in any term, is one less than the number of the term in the progression. Thus, the third term is $a+2d$; hence

$$l = a + d(n-1). \qquad \text{I.}$$

1. Find the 10th term of the progression 2, 5, 8, etc.

By the conditions, $a=2$, $d=3$, $n=10$,

by I, 10th term $=2+3\,(10-1)=29$.

2. Find the 10th term of the progression in which the 3d term is 11, and the 7th term is 27.

By the conditions, $a + 2d = 11,$ (1)
and, $a + 6d = 27,$ (2)
subtracting (1) from (2), $4d = 16,$ (3)
or, $d = 4,$ (4)
substituting $d = 4$, in (1), $a = 3,$ (5)
by I, 10th term $= a + 9d = 39.$ (6)

EXERCISE LXXII

Find the last term of each of the following progressions:

1. 2, 5, 8, ⋯ to 10 terms. **2.** 8, 5, 2, ⋯ to 10 terms.

3. 100, 95, 90, ⋯ to 15 terms.

4. $b, b - c, b - 2c,$ ⋯ to 13 terms.

Find the *n*th term of the following progressions in which:

5. $a = 3\frac{1}{4}, d = 2\frac{5}{6}, n = 10.$

6. $a = 76\frac{3}{5}, d = -4\frac{2}{7}, n = 8.$

7. $a = b + c, d = b - c, n = p.$

8. $a = x - y, d = -y, n = x^2 - y^2.$

Find the indicated terms in the following progressions:

9. 7th term; the 3d being 10, and the 10th, -5.

10. 6th term; the 4th being 0, and the 9th, 15.

11. 1st term; the 7th being -48, and the 13th, -108.

12. 10th term; the 5th being 28, and the 9th, 52.

13. 15th term; the 31st being -40, and the sum of the 3d and 11th, 4.

88. When three quantities are in arithmetical progression, the middle term is called the **arithmetical mean** between the other two.

If a, b, and c are in arithmetical progression, the arithmetical mean b can be found in terms of the other two. Since $b - a = c - b$, $b = \frac{1}{2}(a + c)$.

Hence, *the arithmetical mean between two quantities is one half the sum of the quantities.*

In an arithmetical progression containing any number of terms, all the terms between the first and last are called **arithmetical means** between those terms.

Insert 6 arithmetical means between 8 and 29.

The progression evidently contains 8 terms; $a = 8$, $n = 8$, $l = 29$.

By I, $$29 = 8 + d(8 - 1), \tag{1}$$

solving (1), $$d = 3. \tag{2}$$

Hence the progression is, 8, [11, 14, 17, 20, 23, 26,] 29.

EXERCISE LXXIII

1. Insert 7 arithmetical means between 69 and 95.

2. Insert 13 arithmetical means between 13 and 209.

3. Insert 98 arithmetical means between 6 and -489.

4. Insert 99 arithmetical means between -5780 and 0.

5. Insert 4 arithmetical means between k and $\frac{2k - 3m}{2}$.

6. Insert 10 arithmetical means between $m + \sqrt{3}$ and $m + \sqrt{3} + 729$.

7. Insert r arithmetical means between 1 and $3r - 2$.

89. If S denotes the sum of n terms of an arithmetical progression,

$$S = a + (a + d) + (a + 2d) + \cdots + (l - d) + l, \quad (1)$$

or, $$S = l + (l - d) + (l - 2d) + \cdots + (a + d) + a, \quad (2)$$

adding (1) and (2),

$$2S = (a+l) + (a+l) + (a+l) + \cdots + (a+l) + (a+l), \quad (3)$$

or, $$2S = n(a + l), \quad (4)$$

whence $$S = \frac{n}{2}\left(a + l\right). \quad \text{II.}$$

Since, by I, $l = a + d(n - 1)$, substituting I in II,

$$S = \frac{n}{2}\left\{2a + d(n - 1)\right\}. \quad \text{III.}$$

Equations I, II, and III are called **formulas** of arithmetical progression.

1. Find the sum of 6 terms of the progression, 5, 3, 1, -1, etc.

By the conditions, $a = 5, d = -2, n = 6$.

Substituting a, d, and n in III, $S = \frac{6}{2}\{10 - 2(5)\} = 0$.

2. How many terms of the progression, 4, 7, 10, $\cdots$ must be taken in order that the sum may be 69?

By the conditions, $a = 4$, $d = 3$, $S = 69$.

Substituting a, d, and S in III, $$69 = \frac{n}{2}\left\{8 + 3(n - 1)\right\}, \quad (1)$$

reducing (1), $$3n^2 + 5n - 138 = 0, \quad (2)$$

solving (2), $$n = 6, \text{ or } -\tfrac{23}{3}. \quad (3)$$

Since n must always be a positive integer, $n = 6$ is the only solution.

Problems of the class stated above will evidently always involve the solution of a quadratic equation, and it is therefore possible to obtain one, two, or no correct solutions according as one, two, or no solutions of the quadratic equation are positive integers.

3. In an arithmetical progression whose first term is 3, the sum of 7 terms is 105. Find the common difference.

By the conditions, $a = 3$, $n = 7$, $S = 105$.

Substituting a, n, and S in III, $105 = \frac{7}{2}(6 + 6d)$, (1)

solving (1), $d = 4$.

EXERCISE LXXIV

Find the sum in each of the following progressions:

1. $1, 2, 3, 4, \cdots$ to 10 terms.

2. $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}, \cdots$ to 12 terms.

3. $7, 17, 27, \cdots$ to 8 terms.

4. $2, 2\frac{2}{3}, 3\frac{1}{3}, \cdots$ to m terms.

5. $6\frac{2}{5}, 9\frac{21}{40}, 12\frac{13}{20}, \cdots$ to 13 terms.

6. $100, 90, 80, \cdots$ to 21 terms.

7. $178, 171, 164, \cdots$ to 11 terms.

8. $1, 1 + \sqrt{2}, 1 + 2\sqrt{2}, \cdots$ to r terms.

Find the number of terms in each of the following progressions, so that the given sum may be obtained:

9. $S = 45;\ 15, 12, 9, \cdots.$

10. $S = -1545;\ 50, 43, 36, \cdots.$

11. $S = 1200;\ 31, 38, 45, \cdots.$

12. $S = 52\frac{1}{2};\ \frac{3}{4}, \frac{7}{8}, 1, \cdots.$

13. $S = 30(1 + \frac{7}{6}\sqrt{2});\ 3 - \sqrt{2}, 3, 3 + \sqrt{2}, \cdots.$

In the following arithmetical progressions:

14. Find d, and l, if $a = 3$ and the sum of the first 13 terms is 351.

15. Find d, if the 12th term is 38 and the sum of the first 13 terms is 351.

16. Find d, and l, if $a = 222$, $n = 223$, and $S = 0$.

17. Find a, and l, if $d = 6$, $n = 10$, and $S = 310$.

18. Find n, and d, if $a = 4$, $l = -22$, and $S = -99$.

19. Find n, and d, if $a = \frac{1}{3}$, $l = 15\frac{1}{3}$, and $S = 47$.

20. Find n, and a, if $d = x - 1$, $l = x^3 + x^2 + 3x - 1$, and $S = 3x^3 + 3x^2 + 6x$.

21. The sum of the first 6 terms is 261, and the sum of the first 9 terms is 297. Find the first 9 terms.

22. The sum of the first 3 terms is 14, and the sum of the squares of these terms is 78. Find the terms.

23. The sum of the first half of the terms is 28, the sum of the second half is 222, the sum of the first and last terms is 50. Find the number of terms.

24. The sum of the last four terms is 20, the product of the second and fifth is 16. If the progression contains five terms, find the progression.

25. In a progression of eighteen terms the product of the two middle terms is 90, and the product of the first and eighteenth terms is 18. Find the first and last terms.

GEOMETRICAL PROGRESSION

90. A succession of terms, each of which is obtained from the preceding term by multiplying it by the same positive or negative quantity (the **common ratio**), is called a **geometrical progression.**

Thus, 2, 4, 8, 16, etc., and 1, -3, 9, -27, etc., are geometrical progressions.

The first term is usually represented by a, and the common ratio by r; hence the progression is $a + ar + ar^2 + ar^3$, etc. The number of terms in a progression is represented by n, and the nth term by l.

Since each term is formed from the preceding term by multiplying it by r, the exponent of r in any term is one less than the number of the term. Thus, the third term is ar^2; and the nth term or

$$l = ar^{n-1}. \qquad \text{I.}$$

1. Find the 7th term of the progression 1, -3, 9, ….

By the conditions, $a = 1,\ r = -3,\ n = 7,$

by I, 7th term $= 1\,(-3)^6 = 729.$

2. If the 4th term of a geometrical progression is 1, and the 7th term is $\frac{1}{8}$, find the 1st term.

By the conditions, $ar^3 = 1,$ (1)

and, $ar^6 = \frac{1}{8},$ (2)

dividing (2) by (1), $r^3 = \frac{1}{8},$ (3)

from (3), $r = \frac{1}{2},$ (4)

substituting $r = \frac{1}{2}$ in (1), $a = 8.$ (5)

EXERCISE LXXV

Find the last term in each of the following geometrical progressions:

1. 2, 6, 18, ⋯ to 7 terms.

2. 3, -6, 12, ⋯ to 6 terms.

3. 4, 8, 16, ⋯ to 7 terms.

4. 27, 9, 3, ⋯ to 8 terms.

5. 6, 3, $\frac{3}{2}$, ⋯ to 10 terms.

6. 1, $-\frac{5}{3}$, $\frac{25}{9}$, ⋯ to 11 terms.

In the following geometrical progressions:

7. Find the 7th term, the 2d term being 75, and the 5th, $-\frac{3}{5}$.

8. Find the 2d term, the 4th term being -5, and the 7th, 625.

9. Find the 15th term, the 5th term being $\frac{27}{8}$, and the 10th, $\frac{3^8}{2^8}$.

10. Find the 10th term, the 17th being 1250, and the 13th, 2.

11. Find the 11th term, the 2d term being $b^2 - c^2$, and the 5th, $(b + c)(b - c)^4$.

12. Find the 10th term, the 3d term being b^2, and the 7th, $\frac{b^6}{a^4}$.

13. Find the 7th term, the 2d term being 1, and the 4th, $17 - 12\sqrt{2}$.

14. Find the 8th term, the 4th term being $49 - 20\sqrt{6}$, and the 6th, $485 - 198\sqrt{6}$.

15. Find the 7th term, the 3d term being -2, and the 8th, $-2\,i$.

91. When three quantities are in geometrical progression, the middle term is called the **geometrical mean** between the other two.

If a, b, and c are in geometrical progression, the geometrical mean, which is a mean proportional, can be found in terms of the other two. Since $\frac{b}{a} = \frac{c}{b}$,

$$b^2 = ac, \qquad (1)$$

extracting square roots in (1), $$b = \sqrt{ac}. \qquad (2)$$

Hence, *the geometrical mean between two quantities is the square root of the product of those quantities.*

In a geometrical progression containing any number of terms, all the terms between the first and last are called **geometrical means** between those terms.

Insert 3 geometrical means between 6 and 486.

The progression evidently contains 5 terms; $a = 6$, $n = 5$, $l = 486$.

By I, $$486 = 6r^4, \qquad (1)$$

solving (1), $$r = 3. \qquad (2)$$

Hence the progression is 6, [18, 54, 162,] 486.

EXERCISE LXXVI

1. Insert 2 geometric means between 1 and 64.
2. Insert 6 geometric means between $\frac{a^4}{b^4}$ and $-\frac{b^3}{a^3}$.
3. Insert 11 geometric means between 1 and 2.
4. Insert 5 geometric means between 1875 and 3.
5. Insert 5 geometric means between 36 and $-\frac{2}{243}$.

92. If S denotes the sum of n terms of a geometrical progression,

$$S = a + ar + ar^2 + ar^3 + \cdots ar^{n-2} + ar^{n-1}. \tag{1}$$

Multiplying (1) by r,

$$rS = ar + ar^2 + ar^3 + ar^4 + \cdots ar^{n-1} + ar^n \tag{2}$$

subtracting (1) from (2),

$$S(r-1) = ar^n - a, \tag{3}$$

from (3), $$S = \frac{ar^n - a}{r-1} = \frac{a(r^n - 1)}{r-1}. \tag{II.}$$

Since $l = ar^{n-1}$, $rl = ar^n$, substituting rl for ar^n in II,

$$S = \frac{rl - a}{r-1}. \tag{III.}$$

1. Find the sum of the progression, 2, 6, 18, ⋯ to 6 terms.

By the conditions, $a = 2,\ r = 3,\ n = 6.$

By II, $S = \dfrac{2(3)^6 - 2}{3-1} = 728.$

2. The 3d term of a geometrical progression is 27, the 5th is 81. Find the sum of the first 5 terms.

By the conditions, $ar^2 = 27,$ (1)

and, $ar^4 = 81,$ (2)

dividing (2) by (1), $r^2 = 3,$ (3)

from (3), $r = \sqrt{3},$ (4)

substituting in (1), $a = 9,$ (5)

substituting in II, $S = \dfrac{9(\sqrt{3})^5 - 9}{\sqrt{3} - 1} = 117 + 36\sqrt{3}.$ (6)

EXERCISE LXXVII

In the following geometrical progressions:

1. Find the sum of 3, -6, 12, $\cdots$ to 6 terms.

2. Find the sum of 6, $\frac{3}{2}$, $\frac{3}{8}$, $\cdots$ to 10 terms.

3. Find the sum of $\frac{2}{3}$, $\frac{1}{2}$, $\frac{3}{8}$, $\cdots$ to 10 terms.

4. Find the sum of $\sqrt{2}$, 2, $2\sqrt{2}$, $\cdots$ to 8 terms.

5. Find the sum of $\sqrt{2}+1$, 1, $\sqrt{2}-1$, $\cdots$ to 5 terms.

6. Find the sum of the first 7 terms, if the 2d term is 4, and the 5th, 256.

7. Find the sum of the first 5 terms, if the 3d term is 27, and the 5th, 48.

8. If $a = 6$, and $r = -2$, find n, if the sum of n terms is -30.

9. The sum of the first 5 terms is 242, and the common ratio is 3. Find the 5th term.

10. The sum of the first 4 terms is $9\frac{1}{3}$, and the common ratio is $\frac{1}{3}$. Find the 1st term.

11. Find the sum of the first 6 terms, if the 6th term is $-\frac{40}{243}$ and the common ratio is $-\frac{2}{3}$.

12. Find the common ratio, and the sum of the first 5 terms, if the 1st term is $\frac{2}{3}$ and the 6th term is 864.

13. Find the sum of the first 10 terms of a geometric progression in which the 1st term is 243 and the common ratio is $-\dfrac{1}{\sqrt{3}}$.

14. If the 4th term is $\frac{1}{16}$, and the 7th term is $\frac{1}{128}$, how many terms, beginning with the 1st, must be taken so that their sum is $\frac{4095}{4096}$?

93. If a quantity retains the same value throughout a particular investigation, it is called a **constant.** If a quantity changes in value during a particular investigation, it is called a **variable.**

When the value of a variable can be made to approach the value of a constant in such a way that the difference of the variable and the constant can be made less than any assigned quantity, however small, the constant is called the **limit** of the variable.

SUM OF AN INFINITE GEOMETRICAL PROGRESSION

94. If $r > 1$, each term of a geometrical progression is larger than the preceding term, and the sum of n terms must increase indefinitely as n increases. If $r = 1$, the terms are all equal, and the sum of n terms must again increase indefinitely as n increases. If r is numerically less than unity, each term is less than the preceding term; and it will be shown in the next paragraph that the sum of n terms always remains less than some definite, finite quantity; from which, however, by increasing n, it can be made to differ by less than any assigned quantity, however small.

As an illustration, consider the geometrical progression, $1 + \frac{1}{2} + \frac{1}{4} \cdots$. Applying III, $S = \dfrac{1 - \frac{1}{2} l}{1 - \frac{1}{2}} = 2 - l$. Hence, in this progression, the sum of any number of terms differs from 2 by just the last term. But, by increasing n the last term can be made as small as may be required. Evidently the sum of n terms can never be as large as 2, but it can be made to differ from 2 by a quantity less than any assigned value. Hence 2 is the limit of the sum of n terms, as n increases indefinitely.

95. When $r < 1$, it is convenient to write II in the form,

$$S = \frac{a - ar^n}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

Here r^n can be made as small as is required by increasing n. The second fraction, $\frac{ar^n}{1-r}$, can, therefore, be made as small as is required by increasing the number of terms; and S can be made to differ from $\frac{a}{1-r}$ by less than any assigned quantity. $\frac{a}{1-r}$ is, therefore, the limit approached by S as n increases indefinitely. It is usually called the *sum of the infinite geometrical progression*, but this must always be understood to mean *the limit of the sum of the progression as n increases indefinitely.*

If S represents the limit of that sum,

$$S = \frac{a}{1 - r}. \qquad \text{IV.}$$

1. Find the sum of an infinite number of terms in the progression, $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{18}$, etc.

By IV, $$S = \frac{\frac{1}{2}}{1 - \frac{1}{3}} = \frac{3}{4}.$$

2. Find the value of 0.4545 ⋯.

The decimal 0.4545 is evidently the geometric progression

$$\tfrac{45}{100} + \tfrac{45}{10000} + \tfrac{45}{1000000} + \cdots,$$

in which $$a = \tfrac{45}{100},\ r = \tfrac{1}{100}.$$

By IV, $$S = \frac{\frac{45}{100}}{1 - \frac{1}{100}} = \frac{5}{11}.$$

3. Find the value of 0.4555 ⋯.

The decimal 0.4555 is evidently $\frac{4}{10}$ + the progression

$$\frac{5}{100}+\frac{5}{1000}+\frac{5}{10000}+\cdots,$$

in which $a=\frac{5}{100},\ r=\frac{1}{10}.$

By IV, $$S=\frac{\frac{5}{100}}{1-\frac{1}{10}}=\frac{1}{18}. \tag{1}$$

Hence $$0.4555\cdots=\frac{4}{10}+\frac{1}{18}=\frac{41}{90}. \tag{2}$$

EXERCISE LXXVIII

In the following infinite geometrical progressions:

1. Sum to infinity, $2,\ -\frac{4}{3},\ \frac{8}{9},\ \cdots$.
2. Sum to infinity, $5,\ 2\frac{1}{2},\ 1\frac{1}{4},\ \cdots$.
3. Sum to infinity, $3\frac{3}{8},\ -2\frac{1}{4},\ 1\frac{1}{2},\ \cdots$.
4. Sum to infinity, $4,\ -\frac{4}{3},\ \frac{4}{9},\ \cdots$.
5. Find the value of 0.2544 ⋯.
6. Find the value of 0.86464 ⋯.
7. Find the value of 0.5124545 ⋯.
8. Find the value of 0.2162525 ⋯.
9. Find the value of 0.1248248 ⋯.
10. Find the value of 0.18301830 ⋯.

11. Find the sum to infinity, if the 4th term is 36 and the 7th is $-10\frac{2}{3}$.

12. Find the 1st term, if the sum to infinity is $-1\frac{4}{5}$ and the 2d term is 2.

13. Find the 4th term, if the 1st term is 100 and the sum to infinity is $111\frac{1}{9}$.

CHAPTER XI

PERMUTATIONS AND COMBINATIONS

96. The various orders in which a number of things can be arranged are called their **permutations.**

Thus, a and b can be arranged ab, ba; while a, b, and c can be arranged abc, acb, bac, bca, cab, cba.

97. The various groups that can be selected out of a number of things, without reference to their order, are called their **combinations.**

Thus, the groups of two things that can be selected from a, b, and c are ab, ac, and bc.

Unless the contrary is expressly stated, the things whose permutations or combinations are required will be understood as different things.

Thus, the number of permutations of three different things, when taken two at a time, may be required.

98. *If a single operation can be done in m different ways, and when this operation has been done, if a second operation can be done in n different ways, the two operations can be done together in mn different ways.*

With the first way of performing the first operation there may be associated any one of the n ways of performing the second operation; with the second way of performing the first operation there may be associated any one of the n ways of performing the second operation, etc. That is, with each one of the m different ways of performing the first operation there may be associated n ways of performing the second operation. Therefore there are mn different ways of performing the two operations.

Thus, the offices of president and vice-president can be filled from five candidates in 20 ways; since any one of the five can be selected for president, the office of president can be filled in five different ways; when the office of president has been filled, any one of the remaining four candidates can be selected for vice-president. Any one way of the five ways of filling the office of president can be associated with any one way of the four ways of filling the office of vice-president. Therefore the two offices may be filled in $5 \cdot 4 = 20$ different ways.

Similarly, the above principle applies to more than two operations, each one of which can be performed in a definite number of ways.

Thus, if a man has 5 coats, 3 waistcoats, and 6 pairs of trousers, he can dress himself in $5 \cdot 3 \cdot 6 = 90$ different ways.

PERMUTATIONS

99. *The number of permutations of n different things taken r at a time is* $n(n-1)(n-2) \cdots (n-r+1)$.

The problem of computing the number of permutations of n different things taken r at a time is equivalent to the problem of filling r different places with n different things.

The first place can evidently be filled with any one of the n different things. After the first place has been filled there remain $n-1$ different things, any one of which can be put into the second place; that is, the second place can be filled in $n-1$ different ways for each way that the first can be filled. Hence the first two places can be filled in $n(n-1)$ different ways.

After filling the second place, there remain $n-2$ different things, any one of which can be put into the third place; that is, the third place can be filled in $n-2$ different ways. Hence the first three places can be filled in $n(n-1)(n-2)$ different ways, etc.

Place	1st	2d	3d	4th	...	rth
Number of ways . .	n	$n-1$	$n-2$	$n-3$	...	$n-(r-1)$

Continuing the process, it is evident that the number of ways in which each place can be filled is found by subtracting from n that number which is one less than the number of the place. Hence the rth place can be filled in $n-(r-1)=n-r+1$ different ways. Therefore the r different places can be filled by n different things in $n(n-1)(n-2)\cdots(n-r+1)$ different ways.

The symbol for the number of permutations of n different things taken r at a time is written ${}_nP_r$. Hence

$$ {}_nP_r = n(n-1)(n-2)\cdots(n-r+2)(n-r+1). \qquad \text{I.} $$

100. The number of permutations of n different things taken n at a time can evidently be found by substituting n for r in I,

$$ {}_nP_n = n(n-1)(n-2)\cdots(2)(1). \qquad \text{II.} $$

The product of the factors of ${}_nP_n$, that is, the product of the first n integral numbers, is called **factorial n**, and is written $\underline{|n}$ or $n!$ Formula II may therefore be written

$$ {}_nP_n = n! \qquad \text{II.}$$

EXAMPLES

1. In how many ways can 8 different letters be inserted in 3 different letter-boxes, one and only one being placed in each box?

The first letter-box can be filled in 8 different ways; the second in 7 different ways; the third in 6 different ways; and the three in $8 \cdot 7 \cdot 6 = 336$ different ways. That is,

by I, $$ {}_8P_3 = 8(8-1) \cdots (8-3+1), $$
$$ = 8 \cdot 7 \cdot 6 = 336. $$

2. In how many ways can the letters of the word *Pingry* be arranged?

Since there are 6 different letters, the 6 different letters may be arranged in the 6 different places occupied by the letters in $6!$ different ways; or,

by II, $$ {}_6P_6 = 6! = 720. $$

3. In how many different ways can 5 people be seated at a round table?

The order of arrangement cannot be that of position on a straight line, but on a closed curve. If one of the 5 be seated, so as to give a starting-point from which to reckon the order, the remaining 4 can be seated in the remaining 4 places in $4!$ different ways; or,

by II, $$ {}_4P_4 = 4! = 24. $$

101. *The number of combinations of* n *different things taken* r *at a time is* $\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$.

The symbol for the number of combinations of n different things taken r at a time is written ${}_nC_r$.

Each one of the combinations of ${}_nC_r$ is a selection of r different things which can be arranged, by II, in $r!$ different ways; hence the number of combinations of n different things taken r at a time, or ${}_nC_r$, when multiplied by $r!$ equals the number of permutations of ${}_nP_r$; that is,

$${}_nC_r \cdot r! = n(n-1)\cdots(n-r+1),$$

or

$${}_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}. \qquad \text{III.}$$

The combinations of a, b, and c, taken two at a time, are ab, ac, and bc. Each combination can be arranged in two different ways. Hence ${}_3P_2 = {}_3C_2 \cdot 2$; or

$${}_3C_2 = \frac{{}_3P_2}{2} = \frac{3 \cdot 2}{2} = 3.$$

102. Formula III is employed in obtaining arithmetical results, but the better form of III for algebraic use is

$${}_nC_r = \frac{n!}{r!(n-r)!}. \qquad \text{IV.}$$

Since by III,

$${}_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!},$$

$$= \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} \cdot \frac{(n-r)!}{(n-r)!}$$

$$= \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1}{r!\,(n-r)!}$$

$$= \frac{n!}{r!\,(n-r)!}.$$

By IV, $${}_nC_{n-r} = \frac{n!}{(n-r)!\,[n-(n-r)]!} = \frac{n!}{r!\,(n-r)!}.$$

Therefore, ${}_nC_r = {}_nC_{n-r}$.

EXAMPLES

1. How many committees of 4 men can be formed from 10 men?

Four men are to be selected from 10 men; hence, by III,

$${}_{10}C_4 = \frac{10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4} = 210.$$

2. From 11 men find how many committees of 4 men can be selected, when one man is always included on the committee.

Since one man is always included on the committee, the problem is to select 3 men from the remaining 10; hence, by III,

$${}_{10}C_3 = \frac{10\cdot 9\cdot 8}{1\cdot 2\cdot 3} = 120.$$

3. From 9 men find how many committees of 3 men can be selected, when one man is always excluded from the committee.

Since one man is always excluded from the committee, the problem is to select 3 men from the remaining 8 men; hence, by III,

$${}_8C_3 = \frac{8\cdot 7\cdot 6}{1\cdot 2\cdot 3} = 56.$$

4. How many baseball nines can be made up out of 12 players?

Nine men are to be selected from 12 men; hence, by III and IV,

$$ {}_{12}C_9 = {}_{12}C_3 = \frac{12 \cdot 11 \cdot 10}{1 \cdot 2 \cdot 3} = 220. $$

5. A club consists of 8 seniors and 6 juniors. How many different committees of five can be selected from the club, each committee to consist of 3 seniors and 2 juniors?

By III, the seniors can be chosen in ${}_8C_3 = \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = 56$ ways.

By III, the juniors can be chosen in ${}_6C_2 = \frac{6 \cdot 5}{1 \cdot 2} = 15$ ways.

By § 98, the entire committee can be chosen in $56 \cdot 15 = 840$ different ways.

6. If letters in any order form a word, how many words can be formed from 8 consonants and 5 vowels, each word consisting of 4 consonants and 3 vowels?

By III, the selections of consonants and vowels are respectively ${}_8C_4$ and ${}_5C_3$.

$$ {}_8C_4 = \frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} = 70, $$

$$ {}_5C_3 = \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} = 10. $$

By § 98, the total number of selections of consonants and vowels is $70 \cdot 10 = 700$. Since each of the 700 combinations consists of 7 different letters, each combination can be permuted in $7! = 5040$ different ways. There are $700 \cdot 5040 = 3{,}528{,}000$ different words.

EXERCISE LXXIX

Find the values of :

1. ${}_{10}P_3$.

2. ${}_{11}P_4$.

3. ${}_{12}P_6$.

4. ${}_{7}P_5$.

5. ${}_{7}P_7$.

6. ${}_{n}P_a$.

7. ${}_{6}C_2$.

8. ${}_{7}C_3$.

9. ${}_{8}C_5$.

10. ${}_{12}C_8$.

11. ${}_{12}C_{10}$.

12. ${}_{15}C_{14}$.

13. In how many ways can 4 people be chosen from 10 people?

14. In how many ways can the first 4 letters of the alphabet be arranged?

15. How many numbers of 3 digits each, no digit being repeated, can be formed from the digits 1 to 9 inclusive?

16. In how many different orders can the prime factors of 2310 be written?

17. A man has n different books, which he can place in 5040 different arrangements. Find the number of books.

18. How many combinations can be made of 10 different things taken in sets of 7?

19. On how many nights can a different guard of 5 men be selected from a body of 20? On how many of these would any one man serve?

20. There are 20 things of one kind, and 10 of another. How many different sets can be made each containing 3 of the first kind and 2 of the second?

21. In an examination paper of 10 questions any 3 can be omitted. Find the number of selections.

22. In how many ways can 5 people form a ring? In how many ways a line?

23. How many different committees of 3 Republicans and 3 Democrats can be formed from 10 Republicans and 7 Democrats?

24. How many even numbers of 4 digits each, no digit being repeated, can be formed from the digits 1 to 9 inclusive?

25. In a boat's crew of 8 men one man can row only on stroke side. How many ways can the crew be seated?

26. In how many different ways can a ball nine be arranged, the pitcher and catcher being always the same, but the others playing in any position?

27. How many different sums of money can be formed with a cent piece, a nickel, a dime, a quarter, and a half-dollar?

28. How many different quantities of anything ponderable can be weighed with n different weights?

29. How many changes can be rung with 3 bells out of 6 different bells? How many with the whole peal?

30. From 100 men how many juries of 12 men can be selected if 25 men are excused and if A is always included?

31. If letters in any order form a word, how many words can be formed from 7 consonants and 5 vowels, each word containing 3 consonants and 3 vowels, and ending in a consonant?

32. If each of n straight lines intersects all the others, not more than 2 lines intersecting in the same point, how many points of intersection will there be?

103. *The number of permutations of n different things, taken r at a time,* ***when each of the n things can be repeated, is*** n^r.

After the first place has been filled by one of the n things, the second place can be filled by any one of the n things; and the first two places can be filled in n^2 ways, etc.

Continuing the process, the first three places can be filled in n^3 ways. The exponent of n is evidently the same as the number of places filled. Hence the first r places can be filled in n^r different ways. If x be the number of permutations of n different things, taken r at a time, when each of the n things can be repeated,

$$x = n^r. \qquad \text{V.}$$

104. *The number of permutations of n things, taken n at a time, when p, q, and $r \cdots$ of the n things are respectively a, b, and c, $\cdots$ is* $\dfrac{n!}{p!\,q!\,r!\cdots}$.

The proof will be best understood by taking a specific example: find the number of permutations of $a^3b^2c = a \cdot a \cdot a \cdot b \cdot b \cdot c$.

Place a distinguishing sign on each of the three letters a, and also upon the two letters b, thus: a_1, a_2, a_3, b_1, b_2. Then a_1, a_2, a_3, b_1, b_2, c, are 6 different things which can, by II, be arranged in $6!$ different ways.

Let x be the total number of permutations of a^3b^2c, in which 3 of the letters are a, 2 are b, and 1 is c. Since, by II, the 3 letters a, considered as a_1, a_2, a_3, can be arranged in $3!$ ways, and the 2 letters b can be arranged in $2!$ ways, the total number of permutations of the letters a^3b^2c, considered as different letters, is $x \cdot 3!\,2!$, or $6! = x \cdot 3!\,2!$. Hence $x = \dfrac{6!}{3!\,2!} = 60$.

In general, let x represent the number of permutations of n things, taken n at a time, when $p, q, r, \cdots$ of the n things are respectively $a, b, c, \cdots$. If in any one of the x permutations the p things a were different from each other and all the others, there will be $p!$ different permutations instead of a single permutation. Hence, if all the letters a were changed into p different letters, there would be in all $x \cdot p!$ permutations. Similarly, if in any one of the $x \cdot p!$ permutations, if the q letters b were different from each other and all the others, there would be $x \cdot p!\,q!$ permutations. Continuing the process of changing the letters until they are all different, the total number of permutations will be $x \cdot p!\,q!\,r! \cdots$. Since $n!$ also is the total number of permutations of n different letters, taken n at a time, $n! = x \cdot p!\,q!\,r! \cdots$, or

$$x = \frac{n!}{p!\,q!\,r!\cdots}. \qquad \text{VI.}$$

EXAMPLES

1. Find the number of ways in which a number of 3 digits can be formed of the 9 significant digits, repetitions being allowed.

Each place can be filled in 9 different ways. Hence, by V,

$$x = 9^3 = 729.$$

2. Find the number of arrangements of the letters in the word *Cincinnati*.

Of the 10 letters in the word *Cincinnati*, c occurs twice, i occurs three times, and n occurs three times. Hence, by VI,

$$x = \frac{10!}{2!\,3!\,3!} = 50{,}400.$$

EXERCISE LXXX

1. In how many ways can the following products be written as a different succession of factors: (1) $abcdef$; (2) a^3bc; (3) $a^2b^3c^3$; (4) $a^5b^4c^3$?

2. How many different arrangements can be made of the letters in the following words: (1) *permutation;* (2) *parallel;* (3) *combination;* (4) *Massachusetts;* (5) *incommensurable?*

3. How many words, of 3 letters each, can be formed from a, b, c, e, i, o, u, if repetitions are allowed, and if any order of letters form a word?

4. How many numbers of 3 digits each, repetitions being allowed, can be formed from the first 5 digits?

5. How many odd numbers of 5 digits each, repetitions being allowed, can be formed from 0, 1, 2, ⋯ 9?

6. How many even numbers of 4 digits each, repetitions being allowed, can be formed from the digits 0, 1, ⋯ 9?

7. In how many ways can groups of 4 letters each, repetitions being allowed, be formed from m, n, r, s, u, v, w?

8. In how many ways can groups of 3 letters each be formed from the word *Illinois?*

9. In how many ways can groups of 3 books each be selected from 10 books, 3 of which are the same text in algebra, and 2 of which are the same text in geometry?

10. How many different signals can be formed from 12 flags, 2 being red, 3 green, the rest yellow, if all the flags, placed in line, must be used to make a signal?

CHAPTER XII

PROBABILITY

105. If a certain event may happen in a ways and fail in b ways, and each of these ways are equally likely, the **chance** or **probability** of its happening is $\frac{a}{a+b}$, or *the number of favorable ways divided by the total number of ways it may happen.*

The chance that it may fail is $\frac{b}{a+b}$.

For example, if a single die is thrown, it may turn up in any one of 6 ways and the chance that any particular number shall come up is $\frac{1}{6}$, while the chance that any particular number does not turn up is $\frac{5}{6}$.

The **odds** in favor of the event are $\frac{a}{b}$, or the number of ways it may happen divided by the number of ways it may fail.

The odds against the event are $\frac{b}{a}$.

If the event is certain to happen, its probability is 1; if it is certain to fail, its probability is 0; in all other cases its probability is a positive proper fraction.

EXAMPLES

1. If 2 balls are drawn from a bag containing 3 black and 5 white balls, what is the chance that they both will be black?

Two balls may be drawn ${}_8C_2$ $[=28]$ different ways, while 2 black balls may be selected from the 3 black balls in ${}_3C_2$ $[=3]$ ways.

Hence the chance that the balls will both be black is $\frac{3}{28}$.

2. What is the chance of making a throw of 8 with 2 dice?

There are just $6 \times 6 = 36$ ways in which the two dice may turn up. A throw of 8 will be made only by 6 and 2, 5 and 3, and double 4. But the first two of these may each occur in two ways, so that there are 5 ways in which 8 may be thrown, and the chance that it will occur is $\frac{5}{36}$.

3. If a number of 3 figures is formed from 5 digits, 1, 2, 3, 4, 5, what is the chance that it will contain the digit 1?

The total number of numbers of three figures which can be formed from the five digits is ${}_5P_3 = 5 \times 4 \times 3 = 60$. We must now determine how many of these contain the digit 1. Since one digit is fixed, there is only a choice of two others, and these may be chosen from the four remaining digits. Hence, disregarding their order, we may choose ${}_4C_2$ $[=6]$ sets of three digits which contain 1. But from each of these sets there may be made ${}_3P_3$ $[=6]$ different numbers by changing their order. Hence the total number of numbers which contain 1 is $6 \times 6 = 36$, and the chance that this may occur is $\frac{36}{60} = \frac{3}{5}$.

An alternative method of solving this problem is as follows: We may determine the number of numbers which can be formed, which do *not* contain the digit 1. Since there are 4 other digits, there will be ${}_4P_3$ [$=24$] such numbers. Since 60 numbers may be formed with the 5 digits, the remainder [$=36$] must contain the digit 1. Hence the chance is $\frac{36}{60}$.

4. In a lottery there are 3 prizes and 100 blanks. If a man has two tickets, what is his chance of drawing a prize?

His two tickets may be drawn from the 103 tickets in ${}_{103}C_2$ [$=5253$] different ways. They may both be blank in ${}_{100}C_2$ [$=4950$] ways. Hence there are 303 ways in which he may draw at least one prize, and his chance is

$$\frac{303}{5253}, \text{ or about } \frac{1}{17}.$$

EXERCISE LXXXI

1. If 2 coins are tossed, what is the chance that 2 heads will turn up? 1 head and 1 tail?

2. In a box are 7 white and 3 black balls.

(*a*) If 1 ball is drawn, what is the chance that it will be white? black?

(*b*) If 2 balls are drawn, what is the chance that both will be black? 1 black and 1 white?

(*c*) If 3 balls are drawn, what is the chance that all will be white? 1 white and 2 black?

3. What is the most probable throw with 2 dice, and what is its chance?

4. If 3 dice are thrown, what is the chance that 2 only will be alike?

5. What is the chance of making less than 6 in a throw of 2 dice?

6. If 13 cards are drawn from a complete pack of 52 cards, what is the chance that they will contain 7 spades?

7. If 5 cards are drawn from a pack of 52 cards, what is the chance that they will contain 3 of the same kind? that they will all be of the same suit? that they will contain 4 of the same kind?

8. If 5 coins are tossed, what is the chance that there will be 3 heads and 2 tails?

9. If the chance of an event's happening is $\frac{4}{5}$, what are the odds in its favor?

10. If the odds against an event are $\frac{3}{5}$, what is the chance of its happening?

11. If 13 persons are seated at a round table, show that the odds are 5 to 1 against any 2 particular persons sitting together.

12. If 8 coins are tossed, what is the chance that 1, and only 1, will come up head?

13. In a hand at whist, what is the chance that any particular player should hold the 4 kings?

14. If 10 persons are seated at a round table at random, what is the chance that Miss A will find herself between Mr. B and Mr. C?

15. From a box containing 7 tickets numbered from 1 to 7, 2 tickets are drawn. Find the chance that the product of the numbers on them is even; odd.

CHAPTER XIII

INEQUALITIES

106. The signs > and < express **inequality**: $a>b$ is read "a is greater than b"; $a<b$ is read "a is less than b." Two quantities, a and b, can be compared in three different ways: (1) $a=b$, (2) $a>b$, (3) $a<b$. When $a>b$, $a-b$ is positive; when $a<b$, $a-b$ is negative.

The **first member** of an inequality is the expression to the left of the sign of inequality; and the **second member** is the expression to the right of that sign.

Thus, a^2+b^2 is the first, and $2ab$ the second, member of the inequality, $a^2+b^2>2ab$.

A **term** of an inequality is any term of either the first or second member. Two inequalities **have the same or opposite sense** according as they have, or have not, the same sign of inequality.

Thus, $a>b$ and $c>d$ are inequalities having the same sense; $a>b$, $c<d$, are inequalities which have the opposite sense.

107. The general principles upon which operations with inequalities rest are:

I. *If equals be added to or subtracted from unequals, the resulting inequality has the same sense.*

If $$a>b, \tag{1}$$

then $$a-b>0. \tag{2}$$

Now, $$(a \pm c)-(b \pm c)=a-b, \tag{3}$$

or, substituting (3) in (2),

$$(a \pm c) - (b \pm c) > 0, \tag{4}$$

or, rewriting (4), $$a \pm c > b \pm c. \tag{5}$$

II. *Any term in an inequality may be transposed from one member to the other, if the sign of that term is changed.*

If $$a - c > b, \tag{1}$$

by I, $$a > b + c. \tag{2}$$

III. *If unequals be multiplied or divided by positive equals, the resulting inequality has the same sense.*

If $$a > b, \tag{1}$$

then $$a - b > 0. \tag{2}$$

Let m be any positive integer or fraction. Then $m(a-b)$ must be a positive quantity, since the product of two positive quantities must be positive.

Therefore, $$m(a - b) > 0, \tag{3}$$

or, rewriting (3), $$ma - mb > 0, \tag{4}$$

or, $$ma > mb. \tag{5}$$

IV. *If unequals be multiplied or divided by negative equals, the products have the opposite sense.*

If $$a > b, \tag{1}$$

then $$a - b > 0. \tag{2}$$

Let $-n$ be any negative integer or fraction. Then $-n(a-b)$ must be a negative quantity, since the product of a negative and a positive quantity is a negative quantity.

Therefore, $$-n(a - b) < 0, \tag{3}$$

or, rewriting (3), $$-na + nb < 0, \tag{4}$$

or, $$-na < -nb. \tag{5}$$

Henceforth, *in this chapter, literal quantities are used to represent only positive and unequal quantities.* This fact must be kept in mind, for otherwise the proofs will not hold.

108. A **conditional inequality** is true only for some value or values of the letters involved. An **absolute inequality** is true for all values of the letters involved.

Thus, $2x-3>x+2$ is a conditional, and $a^2+b^2>2ab$ is an absolute, inequality.

EXAMPLES

1. Prove that $a^2+b^2>2ab$.

Either $a-b>0$, or $a-b<0$, and since the square of either a positive or negative number is positive,

$$a^2-2ab+b^2>0,$$

transposing, $a^2+b^2>2ab$.

2. Prove that $a^3+b^3>ab(a+b)$.

As in Ex. 1, $a^2-2ab+b^2>0$,

transposing $-ab$, $a^2-ab+b^2>ab$,

multiplying by $a+b$, $a^3+b^3>ab(a+b)$.

3. Prove that $a^2+b^2+c^2>ab+bc+ca$.

By Ex. 1, $a^2+b^2>2ab$,

and, by Ex. 1, $b^2+c^2>2bc$,

and, by Ex. 1, $c^2+a^2>2ca$,

adding, $2(a^2+b^2+c^2)>2(ab+bc+ca)$,

dividing by 2, $a^2+b^2+c^2>ab+bc+ca$.

4. Prove that $a^3 + b^3 + c^3 > 3\,abc$.

By Ex. 3, $a^2 + b^2 + c^2 > ab + bc + ca,$ (1)

multiplying (1) by a, $a^3 + ab^2 + ac^2 > a^2b + abc + a^2c,$ (2)

multiplying (1) by b, $a^2b + b^3 + bc^2 > ab^2 + b^2c + abc,$ (3)

multiplying (1) by c, $a^2c + b^2c + c^3 > abc + bc^2 + c^2a,$ (4)

adding (2), (3), and (4), and uniting,

$$a^3 + b^3 + c^3 > 3\,abc.$$

5. In the conditional inequality, $3\,x + \frac{4}{3} > x + 8$, find one limit of x.

Let $3\,x + \frac{4}{3} > x + 8.$ (1)

Multiplying (1) by 3, $9\,x + 4 > 3\,x + 24,$ (2)

transposing and uniting in (2), $6\,x > 20,$ (3)

dividing (3) by 6, $x > 3\frac{1}{3}.$ (4)

6. In the conditional inequalities, (1) $x + 7 > \frac{2\,x}{3} + 9$, (2) $\frac{2\,x}{5} < \frac{x}{4} + 2$, find the integral values of x.

Multiplying (1) by 3, $3\,x + 21 > 2\,x + 27,$ (3)

transposing and uniting in (3), $x > 6,$ (4)

multiplying (2) by 20, $8\,x < 5\,x + 40,$ (5)

transposing and uniting in (5), $3\,x < 40,$ (6)

dividing (6) by 3, $x < 13\frac{1}{3}.$ (7)

From (4) and (7), x lies between the limits 6 and $13\frac{1}{3}$; and may therefore take the integral values, 7, 8, 9, 10, 11, 12, 13.

EXERCISE LXXXII

1. Between what limits must x lie, to satisfy the inequalities $2x - 3 > 20$ and $3x - 7 < 2x + 6$?

2. Given $2x - 3 < x + 5$, and $11 + 2x < 3x + 5$, find the limits of x.

3. Given $\frac{2x+3}{2} + 3x < 14$, and $\frac{4x-2}{3} + 2x > 9$, find the limits of x.

4. Given $3x - 5 > 2x + 1$, and $3x + 15 > 4x + 5$, find the limits of x.

Prove the following inequalities, it being understood that the letters represent positive numbers, and that no two of the letters in the same example represent the same number:

5. $\frac{a}{b} + \frac{b}{a} > 2.$

6. $a^2 > 2ab - b^2.$

7. $m^2 + n^2 + p^2 > mn + mp + np.$

8. $a^3b^3 + b^3c^3 + c^3a^3 > 3a^2b^2c^2.$

9. $an + bm < 1$, if $a^2 + b^2 = 1$ and if $m^2 + n^2 = 1$.

10. $ax + by < 15$, if $a^2 + b^2 = 25$ and if $x^2 + y^2 = 5$.

11. $2a^3 + b^3 > a(a^2 + ab + b^2).$

12. $a^3 - b^3 > 3ab(a - b)$, if $a > b$.

13. $(a+b)^3 + (c+d)^3 > (a+b+c+d)(a+b)(c+d)$, if $a + b \neq c + d$.

14. $\frac{m}{p} + \frac{n}{m} + \frac{p}{n} > \frac{n}{p} + \frac{m}{n} + \frac{p}{m}$, if $m > n$, $n > p$, and $m > p$.

CHAPTER XIV

BINOMIAL THEOREM

109. The type forms given in § 4 when $n = 2, 3, 4, 5$, or 6 may be combined into the general form

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2$$

$$+ \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 + \cdots nab^{n-1} + b^n. \qquad \text{I.}$$

A proof of this law — called the **Binomial Theorem** — for the expansion of $(a+b)^n$, when n is any positive integer, will now be given.

1. That I is true when $n = 2, 3, 4, 5$, or 6 may be seen by substituting in I, for example, $n = 3$.

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

If $n = 6$,

$$(a+b)^6 = a^6 + 6a^5b + \frac{6 \cdot 5}{1 \cdot 2}a^4b^2 + \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3}a^3b^3$$

$$+ \frac{6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4}a^2b^4 + \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}ab^5 + \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}b^6$$

$$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6,$$

which is the same result as is obtained by multiplication in § 4.

2. If I is true, when $n = k$, k being any positive integer,

$$(a+b)^k = a^k + ka^{k-1}b + \frac{k(k-1)}{1 \cdot 2}a^{k-2}b^2$$

$$+ \frac{k(k-1)(k-2)}{1 \cdot 2 \cdot 3}a^{k-3}b^3 + \cdots kab^{k-1} + b^k. \qquad (1)$$

3. Multiplying both members of (1) by $a+b$,

$$(a+b)^{k+1} = a^{k+1} + ka^kb + \frac{k(k-1)}{1 \cdot 2}a^{k-1}b^2$$

$$+ \frac{k(k-1)(k-2)}{1 \cdot 2 \cdot 3}a^{k-2}b^3 + \cdots + ab^k$$

$$+ a^kb + ka^{k-1}b^2 + \frac{k(k-1)}{1 \cdot 2}a^{k-2}b^3 + \cdots + kab^k + b^{k+1}$$

$$= a^{k+1} + (k+1)a^kb + \frac{(k+1)k}{1 \cdot 2}a^{k-1}b^2$$

$$+ \frac{(k+1)(k)(k-1)}{1 \cdot 2 \cdot 3}a^{k-2}b^3 + \cdots + (k+1)ab^k + b^{k+1}. \qquad (2)$$

The right member of (2) has the same form as the right member of (1), $(k+1)$ taking the place of k. Hence if the theorem is true for any particular power, it is true for the next higher power.

4. The theorem was shown in 1 to be true for the 6th power; hence it is true for the 7th power; being now true for the 7th power, it is true for the 8th power, and so on for *any* power.

5. The theorem is true for $(a-b)^n$, since $(a-b)^n = [a+(-b)]^n$, the signs of the successive terms being alternately plus and minus, the first term being plus.

NOTE. In the above proof only a few terms of the product are obtained, and to make the proof complete it is necessary to show that the coefficient of $a^r b^{k+1-r}$, any term of (2), has the proper form. This can be done by finding the coefficients of this term in the partial products and adding them.

110. Any required term can be written without completing the expansion by observing the laws for the formation of particular terms. Thus, the fourth term of $(a+b)^n$ is

$$\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3;$$

the rth term is,

$$\frac{n(n-1)(n-2)\cdots(n-r+3)(n-r+2)}{1\cdot 2\cdot 3\cdots(r-2)(r-1)}a^{n-r+1}b^{r-1};$$

and the $(r+1)$st term is,

$$\frac{n(n-1)(n-2)\cdots(n-r+2)(n-r+1)}{1\cdot 2\cdot 3\cdots(r-1)(r)}a^{n-r}b^r.$$

As an aid to the memory the student should note that there is a characteristic number, one less than the number of the term, which belongs to each term; in the $(r+1)$st term there are r factors in both the numerator and denominator of the coefficient, the exponent of a is $n-r$, and the exponent of b is r.

111. The number of terms in the expansion of $(a+b)^n$, when n is a positive integer, is limited. Thus, by I,

$$(a+b)^4 = a^4 + 4a^3b + \frac{4\cdot 3}{1\cdot 2}a^2b^2 + \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3}ab^3$$
$$+\frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4}b^4 + \frac{4\cdot 3\cdot 2\cdot 1\cdot 0}{1\cdot 2\cdot 3\cdot 4\cdot 5}a^{-1}b^5 + \cdots.$$

Since the coefficients of all terms following the fifth contain a zero factor, all such terms disappear. In general, if n is a positive integer, the expansion of $(a+b)^n$ ends with the $(n+1)$st term.

The coefficients of terms equally distant from the end terms are equal. It is evident that

$$(a+b)^n=(b+a)^n.$$

$$(b+a)^n=b^n+nb^{n-1}a+\frac{n(n-1)}{1\cdot 2}b^{n-2}a^2+\cdots nba^{n-1}+a^n. \quad (4)$$

$(b+a)^n$ is merely the expansion of $(a+b)^n$ written in descending powers of b. The last term of I is the same as the first term of (4); the second term of I is the second from the last of (4), etc.

Hence in the expansion of a binomial, *the coefficients of terms equally distant from the end terms are equal.*

EXAMPLES

1. Expand $(3a-1)^5$.

By I,

$$(3a-1)^5=(3a)^5+5(3a)^4(-1)+\frac{5\cdot 4}{1\cdot 2}(3a)^3(-1)^2$$

$$+\frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}(3a)^2(-1)^3+\frac{5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4}(3a)(-1)^4$$

$$+\frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4\cdot 5}(-1)^5$$

$$=243a^5-405a^4+270a^3-90a^2+15a-1.$$

2. Find the 6th term of $\left(1-\frac{\sqrt{b}}{2}\right)^{11}$.

By § 110, the 6th term of

$$\left(1-\frac{\sqrt{b}}{2}\right)^{11}=\frac{11\cdot 10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4\cdot 5}(1)^6\left(-\frac{\sqrt{b}}{2}\right)^5$$

$$=-\frac{3\cdot 7\cdot 11\,b^2\sqrt{b}}{2^4}$$

$$=-\frac{231\,b^2\sqrt{b}}{16}.$$

EXERCISE LXXXIII

Expand the following binomials:

1. $(a-b)^5$.

2. $(a-b^2)^5$.

3. $(a^2+b^2)^4$.

4. $(1+x^2)^7$.

5. $(ab-1)^{11}$.

6. $(x^{-1}+y^{-2})^5$.

7. $(a^3-b^3)^7$.

8. $(2x-y)^5$.

9. $(a-2x)^6$.

10. $(3x-2y)^5$

11. $\left(\frac{1}{x}-2xy\right)^6$.

12. $\left(\frac{2}{a}+\frac{b}{3x}\right)^5$.

13. $\left(\frac{6a}{7}-1\right)^4$.

14. $(2\sqrt{a}-1)^5$.

15. $\left(3a^{\frac{1}{2}}+\frac{1}{b}\right)^6$.

16. $\left(\frac{2a^{-1}}{\sqrt{b}}+\frac{a\sqrt{ab}}{2}\right)^6$.

17. $\left(\frac{2a\sqrt{b}}{a^{-\frac{3}{4}}}-\frac{1}{\sqrt[3]{b}}\right)^5$.

18. $\left(\frac{2a}{x^2}-\frac{x\sqrt{a}}{2}\right)^6$.

19. $\left(\frac{2a^{-1}}{\sqrt{b}}+\frac{a\sqrt{ab}}{2}\right)^7$.

20. $\left(\frac{2a\sqrt{b^{-3}}}{\sqrt[3]{xy}}-\frac{\sqrt{x^3y^3}}{b}\right)^5$.

21. $\left(\frac{2\sqrt{-a}}{b}-\frac{b}{3\sqrt{-a}}\right)^6$.

Express in simplest form the indicated terms of the following binomials:

22. 4th term of $(x-y)^{27}$. **23.** 2d term of $(x-y)^{51}$.

24. 11th term of $(a-b)^{12}$.

25. 5th term of $\left(a^3b-\dfrac{3\,b^{-2}}{\sqrt{a^5}}\right)^{31}$.

26. 6th term of $\left(\dfrac{6\,a^2}{7\,b\sqrt{b}}-\dfrac{b}{\sqrt{3\,a}}\right)^7$.

27. 8th term of $\left(\dfrac{\sqrt[3]{a}}{b}-\dfrac{b^2}{2\,a}\right)^{29}$.

28. 10th term of $\left(\dfrac{9\,a}{\sqrt{b}}-\dfrac{2\,b}{\sqrt{a}}\right)^{27}$.

29. 6th term of $\left(\dfrac{2\sqrt{a}}{3}-\dfrac{6\sqrt[3]{b^2}}{a}\right)^{21}$.

30. 8th and 11th terms of $\left(\dfrac{a\sqrt{a}}{\sqrt[9]{b^2}}-6\sqrt{b^3}\right)^{17}$.

31. 4th and 17th terms of $\left(\dfrac{3\sqrt{a}}{\sqrt[4]{2}\,\sqrt[11]{b^3}}-\dfrac{\sqrt{2}}{3\,a^4\sqrt[10]{b}}\right)^{27}$.

32. $(r+1)$st term of $(2\,a-b)^{2r}$.

33. $(k-2)$d term of $\left(a-\dfrac{2}{\sqrt{b}}\right)^{k+1}$.

34. Find the first 4 and the last 4 terms of $(\sqrt{a}-2\sqrt[3]{b})^{20}$.

35. Find the first 3 and the last 3 terms of $\left(1-\dfrac{2\sqrt{a}}{\sqrt[3]{b}}\right)^{32}$.

CHAPTER XV

LOGARITHMS

112. The **logarithm** of any number is the exponent indicating the power to which a certain fixed number, called the **base**, must be raised in order to produce the given number.

The symbol $\log_b n$ means the logarithm of n with respect to the base b, or if $n = b^l$, then $l = \log_b n$.

EXAMPLES

1. Find the logarithm of 25, if the base is 5.

Since $25 = (5)^2$, $\log_5 25 = 2$.

2. Find the logarithm of 243, if the base is 9.

Since $243 = (3)^5 = (3^2)^{\frac{5}{2}} = (9)^{\frac{5}{2}}$, $\log_9 243 = \frac{5}{2}$.

3. Find the logarithm of $\frac{1}{27}$, if the base is 3.

Since $\dfrac{1}{27} = \dfrac{1}{(3)^3} = (3)^{-3}$, $\log_3 \frac{1}{27} = -3$.

EXERCISE LXXXIV

Find the logarithms of the following numbers :

1. $8, 32, 2\sqrt{2}, \frac{1}{8}, \frac{1}{128}$, the base being 4.
2. $3, 27, 81\sqrt{3}, \frac{1}{9}, \frac{1}{81}$, the base being 9.
3. $2, \frac{1}{8}, 2^{32}, \dfrac{1}{2\sqrt{2}}$, the base being 16.

113. The logarithms of numbers which are not exact powers of the base can only be expressed approximately. Thus there is no power of 10 which is exactly 15. But $10^{\frac{117}{100}}$ is a little less than 15, and $10^{\frac{118}{100}}$ is greater than 15. We may then say that $\log_{10} 15$ is between 1.17 and 1.18. This process may be continued until we know accurately as many places in the logarithm as may be desired. Hence all operations with logarithms are approximate, but the approximation may be carried to as many decimal places as we please. The methods by which logarithms are obtained can not be described here, but we shall assume that we may obtain as accurately as we please the power to which the base must be raised to give any required number. When, therefore, we write $\log_{10} 15 = 1.176$, it is not an absolute equality,. but only as close an approximation as we may need in the problem under discussion.

PRINCIPLES OF LOGARITHMS

114. I. *The logarithm of the product of two or more factors is the sum of the logarithms of the factors.*

Let $b^x = m$, or $\log_b m = x$, (1)

and let $b^y = n$, or $\log_b n = y$, (2)

multiplying (1) and (2), $b^{x+y} = mn$,

or $\log_b mn = x + y = \log_b m + \log_b n$. (3)

Similarly, theorem I can be proved for the product of three or more factors.

II. *The logarithm of the quotient of two numbers is the logarithm of the dividend minus the logarithm of the divisor.*

Let $b^x = m$, or $\log_b m = x$, (1)

and let $b^y = n$, or $\log_b n = y$, (2) dividing (1) by (2), $$b^{x-y}=\frac{m}{n},$$

or $$\log_b \frac{m}{n}=x-y=\log_b m-\log_b n. \qquad (3)$$

III. *The logarithm of the power of a number is the product of the logarithm of the number by the exponent of the power.*

Let $$b^x=n, \text{ or } \log_b n=x, \qquad (1)$$

raising both members of (1) to the kth power,

$$b^{kx}=n^k,$$

or $$\log_b n^k=kx=k\log_b n. \qquad (2)$$

IV. *The logarithm of the root of a number is the quotient obtained by dividing the logarithm of the number by the index of the root.*

Let $$b^x=n, \text{ or } \log_b n=x, \qquad (1)$$

extracting the kth root of both members of (1),

$$b^{\frac{x}{k}}=n^{\frac{1}{k}},$$

or $$\log n^{\frac{1}{k}}=\frac{x}{k}=\frac{\log_b n}{k}. \qquad (2)$$

115. In the **common (or Briggs) System,** the number **10** is always taken as the base. It may be shown that

$$\begin{array}{ll}
10^0=1, & 10^0=1, \\
10^1=10, & 10^{-1}=\frac{1}{10}=0.1, \\
10^2=100, & 10^{-2}=\frac{1}{10^2}=0.01, \\
10^3=1000, & 10^{-3}=\frac{1}{10^3}=0.001, \\
10^4=10000. & 10^{-4}=\frac{1}{10^4}=0.0001.
\end{array}$$

116. Log $1 = 0$ is a short way of writing that, in the system in which the base is 10, the exponent of the power of 10, which produces 1, is 0. Hence,

$$\begin{array}{rr} \log 1 = 0, & \log 1 = 0, \\ \log 10 = 1, & \log 0.1 = -1, \\ \log 100 = 2, & \log 0.01 = -2, \\ \log 1000 = 3, & \log 0.001 = -3, \\ \log 10000 = 4. & \log 0.0001 = -4. \end{array}$$

117. It is evident that a number between 1 and 10 has a logarithm between 0 and 1; a number between 10 and 100 has a logarithm between 1 and 2; a number between 100 and 1000 has a logarithm between 2 and 3; a number between 1 and 0.1 has a logarithm between 0 and -1; a number between 0.1 and 0.01 has a logarithm between -1 and -2; a number between 0.01 and 0.001 has a logarithm between -2 and -3, etc. In general, the logarithm of a number greater than 1 is positive, and the logarithm of a number between 0 and 1 is negative.

118. The logarithm of a number, not an exact power of 10, consists of two parts, — the **characteristic**, which is the integral part, and the **mantissa**, which is a fractional part expressed as a decimal.

The characteristic of the logarithm of any number greater than 1 is always positive, and depends upon the number of digits in the number to the left of the decimal point. From the table in the preceding paragraph, it may be seen that any number containing two digits to the left of the decimal point has a characteristic of 1; that any number containing three digits to the left of the decimal point has a characteristic of 2, etc. Hence:

The characteristic of the logarithm of any number greater than 1 *is always one less than the number of digits preceding the decimal point.*

119. The mantissa of the logarithm of any number is given in the table on pages 204–205. That its value does not depend on the position of the decimal point is shown by the following problem:

Let $\log 214 = 2.3304.$

Then $\log 21.4 = \log (214 \div 10) = 2.3304 - 1$

$= 1.3304,$

$\log 2.14 = 1.3304 - 1 = .3304,$

$\log .214 = .3304 - 1.$

Here it would be inconvenient to find the logarithm by subtracting .3304 from 1. It is, therefore, written in the form,

$$9.3304 - 10.$$

Again, $\log .0214 = 8.3304 - 10,$

$\log .00214 = 7.3304 - 10,$

$\log .000214 = 6.3304 - 10,$ etc.

Hence *the mantissas of the logarithms of all numbers are positive and are the same for all numbers which have the same sequence of digits, without reference to the position of the decimal point.*

It also appears from the above example that *the characteristic of any number between zero and unity is negative, and in absolute value one more than the number of zeros after the decimal point.*

These negative characteristics are always written as in the example, 9 − 10, 8 − 10, etc.

USE OF THE TABLE

120. To find the logarithm of a number consisting of three digits:

On pages 204–205 *find in the column under N the first two digits of the given number. The mantissa required will be found at the intersection of the horizontal line containing the first two digits and the vertical column headed by the third digit. Prefix the proper characteristic.*

$$\log 21.7 = 1.3365,$$
$$\log 0.429 = 9.6325 - 10,$$
$$\log 970 = 2.9868,$$
$$\log 0.0211 = 8.3243 - 10.$$

Numbers containing less than three digits are similarly found.

$$\log 0.27 = 9.4314 - 10,$$
$$\log 5 = 0.6990,$$
$$\log 0.0029 = 7.4624 - 10.$$

121. To find the logarithm of a number consisting of more than three digits:

1. Find the logarithm of 92.04.

Mantissa of the log of the sequence $920 = 9638$,

mantissa of the log of the sequence $921 = 9643$.

An increase of one unit in the sequence gives an increase of 0.0005 in the mantissa; an increase of 0.4 of a unit in the sequence gives an increase of $0.4 \times 0.0005 = 0.0002$ in the mantissa. Therefore,

mantissa of the log of the sequence $9204 = 9640$,

prefixing required characteristic, $\log 92.04 = 1.9640$.

2. Find the logarithm of 0.01238.

Mantissa of the log of the sequence $123 = 0899$,

mantissa of the log of the sequence $124 = 0934$.

An increase of one unit in the sequence gives an increase of 0.0035 in the mantissa; an increase of 0.8 of a unit in the sequence gives an increase of $0.8 \times 0.0035 = 0.0028$ in the mantissa. Therefore

mantissa of the log of the sequence $1238 = 0927$,

prefixing required characteristic, $\log\ 0.01238 = 8.0927 - 10$.

122. The process of making the proper correction in the logarithms of numbers of more than three digits is called **Interpolation**, and is based upon the hypothesis that adjacent mantissas increase proportionally with the corresponding numbers. Corrections made in this manner are not strictly accurate; and even the mantissas given are only approximate, but are correct to 0.00005. If the correction in the fifth decimal place be 5 or more, the fourth decimal place is increased by 1.

In the table on pages 204–205 *find the mantissa of the first three significant digits, disregarding the position of the decimal point; subtract the mantissa thus found from the mantissa of the next higher number of three significant digits; multiply the difference thus found by the decimal represented by the remaining digits of the given number; add the product (to the fourth decimal) to the mantissa of the first three digits. Prefix the proper characteristic.*

123. To find the number corresponding to a given logarithm.

1. Find the number whose logarithm is 7.5521 − 10.

From the table, 5514 is the mantissa of the sequence 356, and 5527 is the mantissa of the sequence 357; that is, a difference of 0.0013 in the mantissa gives a difference of one unit in the sequence; hence the mantissa 5521, being 0.0007 more than the mantissa 5514, gives a difference of $\frac{7}{13}$ of one unit ($=0.5$) in the sequence. Therefore, applying § 119,

$$\log 0.003565 = 7.5521 - 10.$$

The number corresponding to a given logarithm is called the ***antilogarithm.***

EXERCISE LXXXV

Find the logarithms of the following numbers:

1. 254.
2. 465.
3. 200.
4. 908.
5. 2.
6. 20.
7. 362.
8. 5685.
9. 6297.
10. 1004.
11. 0.8562.
12. 0.003547.
13. 8.437.
14. 0.003.
15. 0.000569.
16. 0.009186.
17. 0.01089.
18. 0.9989.

Find the antilogarithms of:

19. 0.3927.
20. 1.6395.
21. 3.7235.
22. 9.8420 − 10.
23. 7.9069 − 10.
24. 6.9903 − 10.
25. 0.9321.
26. 1.6872.
27. 3.5589.
28. 5.6372.
29. 4.3204.
30. 2.3974.
31. 0.0250.
32. 9.5299 − 10.
33. 8.7467 − 10.
34. 2.8837.
35. 8.9432 − 10.
36. 7.0161 − 10.

USE OF LOGARITHMS WHICH HAVE NEGATIVE CHARACTERISTICS

124. *The following problems illustrate the various operations with logarithms having negative characteristics.*

EXAMPLES

1. Add the following logarithms:

$$\begin{array}{l}
9.6253-10\\
\underline{8.5145-10}\\
18.1398-20=8.1398-10.
\end{array}$$

2. Subtract the logarithm 3.1461 from the logarithm 2.1430.

$$\begin{array}{rl}
2.1430= & 12.1430-10\\
\underline{3.1461=} & \underline{3.1461}\\
 & 8.9969-10.
\end{array}$$

3. Subtract the logarithm 9.3141 − 10 from the logarithm 8.6537 − 10.

$$\begin{array}{rl}
8.6537-10= & 18.6537-20\\
\underline{9.3141-10=} & \underline{9.3141-10}\\
 & 9.3396-10.
\end{array}$$

4. Multiply the logarithm 8.1461 − 10 by 2.

$$\begin{array}{r}
8.1461-10\\
\underline{2}\\
16.2922-20=6.2922-10.
\end{array}$$

5. Divide the logarithm 7.9101 − 10 by 3.

$$7.9101-10=27.9101-30$$

$$3\overline{)27.9101-30}$$
$$9.3034-10.$$

In multiplying a logarithm by a fraction, multiply the logarithm by the numerator and divide this product by the denominator, in the order stated, taking care to simplify at each step.

6. Multiply the logarithm 8.3196 – 10 by $\frac{2}{3}$.

$$\begin{array}{r} 8.3196-10 \\ 2 \\ \hline 16.6392-20 = 26.6392-30 \end{array}$$

$$3\overline{)26.6392-30} \qquad 8.8797-10.$$

EXERCISE LXXXVI

Perform the indicated operations in the following logarithms:

1. $(9.7305-10)+(9.3457-10)$.
2. $(8.5478-10)+(9.8438-10)$.
3. $(0.6544)+(9.7253-10)$.
4. $(0.8733)-(2.7459)$.
5. $(9.3476)-(9.5244)$.
6. $(8.2386-10)\times 5$.
7. $(8.8300-10)\div 3$.
8. $(9.1436-10)\times 4$.
9. $(6.8433-10)\times \frac{2}{3}$.
10. $(9.8010-10)\div \frac{4}{5}$.
11. $(7.1431-10)\times \frac{4}{3}+(8.7153-10)$.
12. $(2.5157)\times \frac{1}{3}-(9.9918-10)$.
13. $(6.5000)-(8.5431)\times \frac{2}{5}$.
14. $(7.2511-10)+(8.2190)\times \frac{5}{3}$.
15. $(9.0909)\times 5-(8.1650)\times \frac{7}{4}$.
16. $(2.0001)\times \frac{3}{5}-(8.0999)\times \frac{1}{6}$.

COMPUTATIONS BY LOGARITHMS

125. 1. Find the value of $\dfrac{192.7 \times 6.54 \times 0.4683}{1624 \times 0.0329 \times 1.028}$.

$$\begin{aligned}
\log 192.7 &= 2.2849 \\
\log 6.54 &= 0.8156 \\
\log 0.4683 &= \underline{9.6705-10} \\
\log \text{numerator} &= 2.7710 \\
\log \text{denominator} &= \underline{1.7398} \\
\log \text{fraction} &= 1.0312 \\
\text{fraction} &= 10.75.
\end{aligned}$$

$$\begin{aligned}
\log 1624 &= 3.2106 \\
\log 0.0329 &= 8.5172-10 \\
\log 1.028 &= \underline{0.0120} \\
\log \text{denominator} &= 1.7398
\end{aligned}$$

2. Find the value of $\sqrt{32.5 \times 68.7 \times 32.74}$.

$$\begin{aligned}
\log 32.5 &= 1.5119 \\
\log 68.7 &= 1.8370 \\
\log 32.74 &= \underline{1.5151} \\
\log \text{product} &= 4.8640 \\
\tfrac{1}{2} \log \text{product} &= 2.4320 \\
\text{square root of product} &= 270.4.
\end{aligned}$$

3. Find the value of $(5.235)^3$.

$$\begin{aligned}
\log 5.235 &= 0.7189 \\
3 \log 5.235 &= 2.1567 \\
(5.235)^3 &= 143.5.
\end{aligned}$$

4. Find the value of $-\sqrt{8} \times \sqrt[3]{\frac{1}{7}}$.

$$\begin{aligned}
\log 8 &= 0.9031 \\
\tfrac{1}{2} \log 8 &= 0.4516 \\
\tfrac{1}{3} \log \tfrac{1}{7} &= \underline{9.7183-10} \\
\log \text{product} &= 0.1699 \\
\text{product} &= -0.1479.
\end{aligned}$$

$$\begin{aligned}
\log 1 &= 10.0000-10 \\
\log 7 &= \underline{0.8451} \\
\log \tfrac{1}{7} &= 29.1549-30 \\
\tfrac{1}{3} \log \tfrac{1}{7} &= 9.7183-10.
\end{aligned}$$

Note that the product is negative in accordance with the law of signs.

EXERCISE LXXXVII

Compute by the use of logarithms:

1. 21.4×9.87.
2. 6.92×53.4.
3. 0.908×201.
4. 65.31×0.319.
5. 0.8642×589.7.
6. 0.9034×0.00154.
7. $698 \div 20$.
8. $0.583 \div 2982$.
9. $0.9085 \div 9.805$.
10. $0.9651 \div 0.8939$.
11. $251.2 \div 0.785$.
12. $0.09891 \div 0.001234$.
13. $200.9 \div 10.01$.
14. $8957 \div 0.9081$.
15. $0.7154 \div 9.003$.
16. $0.2167 \div 0.0375$.
17. $0.04678 \div 892$.
18. $0.0001 \div 894.5$.
19. $8.9 \times 0.32 \times 0.065$.
20. $0.8 \times 3 \times 500$.
21. $0.3 \times 0.09 \times 0.1986$.
22. $6.98 \times 0.6851 \times 0.32$.
23. $0.91 \times 0.81 \times 0.09$.
24. $\dfrac{215.4 \times 89.72 \times 0.896}{0.6671 \times 19.2 \times 88.32}$.
25. $\dfrac{6456 \times 0.6456 \times 0.06456}{27 \times 270 \times 2700}$.
26. $\dfrac{0.4692 \times 9231 \times 6482}{0.1492 \times 0.8351 \times 6987}$.
27. $\dfrac{0.5533 \times 419.2 \times 0.3265}{60.90 \times 5.432 \times 0.8406}$.
28. $\dfrac{6384 \times 0.0987 \times 0.012}{2007 \times 0.3388 \times 0.871}$.
29. $(6.608)^2$.
30. $(2.755)^2$.
31. $(1.01)^{25}$.
32. $(99.81)^3$.
33. $\sqrt{64.91}$.
34. $\sqrt{9.181}$.
35. $\sqrt[3]{0.0182}$.
36. $\sqrt{6503}$.
37. $\sqrt[3]{0.6608}$.
38. $\sqrt{\dfrac{23 \times 75}{13 \times 0.85}}$.
39. $\sqrt{\dfrac{0.525 \times 0.054}{351 \times 0.062}}$.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	2677
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

NATURAL LOGARITHMS

126. Systems of logarithms with other numbers than 10 as a base might be constructed, but no other base gives a compact and useful table for actual computation. But for theoretical purposes in certain branches of mathematics it is convenient to use a different system. It is called the **natural** system of logarithms, and the base used is an irrational number 2.718 ⋯, which is always represented by the letter e. Tables of natural logarithms may be constructed, but, since a change in the position of the decimal point changes the entire logarithm, such a table would not be practical. But the logarithm of any number to any base may be obtained from a table of common logarithms by a simple formula which we shall now prove.

Let n be any number, and let x represent its logarithm with respect to any base b.

Then $$x = \log n,$$

and $$n = b^x.$$

Equating the common logarithms of these two equal quantities,

$$\log_{10} n = \log_{10} b^x = x \log_{10} b.$$

Hence $$x = \log_b n = \frac{\log_{10} n}{\log_{10} b}.$$

Hence the rule: *To obtain the logarithm of any number to any base, divide the common logarithm of the number by the common logarithm of the base.*

The common logarithm of e is .4343. Hence the natural logarithm of any number may be obtained by dividing its common logarithm by .4343, or multiplying it by 2.3026, the reciprocal of .4343.

Ex. 1. Find the natural logarithm of 100.

$$\log_e 100 = \frac{\log_{10} 100}{\log_{10} e} = \frac{2}{.4343} = 4.6052.$$

Ex. 2. Find $\log_3 5$.

$$\log_3 5 = \frac{\log_{10} 5}{\log_{10} 3} = \frac{.6990}{.4771} = 1.465.$$

EXPONENTIAL EQUATIONS

127. Equations with variable exponents may be solved by the use of logarithms. The method employed is shown in the following example:

Ex. Find the value of x, if $4^x = 13$.

Equating the logarithms of the equal quantities,

$$\log 4^x = \log 13.$$

Hence $$x \log 4 = \log 13,$$

and $$x = \frac{\log 13}{\log 4} = \frac{1.1139}{.6021} = 1.85\cdots.$$

EXERCISE LXXXVIII

1. Form a table of the natural logarithms of the integers from 1 to 10.

2. Find the natural logarithm of .3, 1.5.

3. Find the logarithm of 45, in a system of which the base is 12.

4. Find $\log_2 10$, $\log_{16} 25$, $\log_6 7$, $\log_5 2.43$.

Solve the following equations:

5. $3^x = 16$.
6. $5^x = 10$.
7. $16^x = 45$.
8. $7^{x+1} = 5$.
9. $31^{x+3} = 25^{x+4}$.
10. $(\frac{1}{3})^x = \frac{1}{25}$.

CHAPTER XVI

PARTIAL FRACTIONS

128. It is sometimes convenient to be able to express an algebraic fraction whose denominator is of higher degree than the first as the sum of several fractions whose denominators are factors of the denominator of the given fraction. The proof of the theorem that this is always possible is rather complicated and will not be given. We shall simply describe the process by which any fraction may be separated into *partial fractions*, whose sum is equal to the given fraction.

In any of the following cases, if the numerator is of as high or higher degree than the denominator, divide the numerator by the denominator until the remainder is of lower degree than the denominator, and separate into partial fractions the fraction formed by writing the remainder over the denominator.

CASE I

129. Factors of the denominator all linear and unequal.

Let the factors of the denominator be

$$(x-a)(x-b)\cdots(x-k).$$

Then the partial fractions are

$$\frac{A}{x-a}+\frac{B}{x-b}+\cdots+\frac{K}{x-k},$$

where A, B, $\cdots$ K are constant quantities. The method of determining the values of these numerators is best shown by an example.

Ex. 1. Separate into partial fractions $\dfrac{3x+7}{(x-1)(x-2)}$.

Let $$\frac{3x+7}{(x-1)(x-2)} \equiv \frac{A}{x-1} + \frac{B}{x-2}.$$

Then $$3x+7 \equiv A(x-2) + B(x-1) \qquad (1)$$
$$\equiv (A+B)x - 2A - B.$$

Since this is an identity, the coefficients of like powers of x must be equal.

Hence $A + B = 3$, and $2A + B = -7$.

Solving, $A = -10$, and $B = 13$.

Hence $$\frac{3x+7}{(x-1)(x-2)} = \frac{13}{x-2} - \frac{10}{x-1}.$$

Another method of obtaining the values of A and B which is often easier than the above is as follows:

Since (1) is an identity, it will hold for all values of x. If we let $x = 1$, equation (1) becomes $A = -10$. If we let $x = 2$, (1) becomes $B = 13$.

The assumption that the given fraction can be separated into two fractions of the form used is justified by the result, since we have obtained two fractions whose sum is the given fraction. If there had not been sufficient equations to determine A and B, or if these equations had not been solvable, it would have been necessary to have changed the assumption, but the student will discover that it is always possible to determine the values of $A, B, \cdots K$.

Ex. 2. Separate into partial fractions

$$\frac{2x^3 + 9x^2 + 8x - 18}{x^3 + x^2 - 6x}.$$

By division $$\frac{2x^3 + 9x^2 + 8x - 18}{x^3 + x^2 - 6x} = 2 + \frac{7x^2 + 20x - 18}{x^3 + x^2 - 6x}.$$

Let $$\frac{7x^2+20x-18}{x^3+x^2-6x} \equiv \frac{A}{x}+\frac{B}{x+3}+\frac{C}{x-2}.$$

Then
$$A(x+3)(x-2)+Bx(x-2)+Cx(x+3) \equiv 7x^2+20x-18.$$

If we let $x=0$, $-6A=-18$, or $A=3$.
If we let $x=-3$, $15B=-15$, or $B=1$.
If we let $x=2$, $10C=50$, or $C=5$.

Hence the given fraction has been separated into the partial fractions, $$2+\frac{3}{x}-\frac{1}{x+3}+\frac{5}{x-2}.$$

EXERCISE LXXXIX

Separate each of the following fractions into partial fractions:

1. $\dfrac{5x+6}{(x+2)(x-1)}$.
2. $\dfrac{x+2}{x^2-5x+6}$.
3. $\dfrac{x}{(x+1)(x+3)(x+5)}$.
4. $\dfrac{10}{1-x^2}$.
5. $\dfrac{1}{x^2-7x+12}$.
6. $\dfrac{x^2+2x-1}{9x^2-16}$.
7. $\dfrac{5x-4}{(x-3)(x^2-1)}$.
8. $\dfrac{x^3-6x+4}{x^2-8x+7}$.
9. $\dfrac{x^2+5x+10}{(x+1)(x+2)(x+3)}$.
10. $\dfrac{x^3-3x^2+4x+14}{x^2+x-6}$.

CASE II

130. Factors of the denominator all linear but some of them equal.

For every factor, $(x-a)$, of the denominator which occurs only once, write a partial fraction of the form $\frac{A}{x-a}$; for *every factor,* $(x-b)$, *which occurs n times, write partial fractions of the form*

$$\frac{B_n}{(x-b)^n}+\frac{B_{n-1}}{(x-b)^{n-1}}+\cdots+\frac{B}{(x-b)}.$$

The latter assumption is justified by the fact that the addition of such a set of partial fractions will give a fraction whose denominator is $(x-b)^n$ and whose numerator is of lower degree than n. It may, of course, be found that any of the numerators except B_n are zero.

Ex. Separate into partial fractions $\dfrac{3x-1}{x^2(x+1)^2}$.

Let $\dfrac{3x-1}{x^2(x+1)^2}\equiv\dfrac{A}{x^2}+\dfrac{B}{x}+\dfrac{C}{(x+1)^2}+\dfrac{D}{x+1}$.

Then
$$\begin{aligned}3x-1&\equiv A(x+1)^2+Bx(x+1)^2+Cx^2+Dx^2(x+1)\\&\equiv(B+D)x^3+(A+2B+C+D)x^2\\&\quad+(2A+B)x+A.\end{aligned}$$

Since this is an identity, the coefficients of like powers of x must be equal.

Hence
$$\begin{aligned}B+D&=0,\\A+2B+C+D&=0,\\2A+B&=3,\\A&=-1.\end{aligned}$$

Solving, $A=-1$, $B=5$, $C=-4$, $D=-5$.

Hence $$\frac{3x-1}{x^2(x+1)^2}=-\frac{1}{x^2}+\frac{5}{x}-\frac{4}{(x+1)^2}-\frac{5}{x+1}.$$

EXERCISE XC

Separate each of the following fractions into partial fractions:

1. $\dfrac{x-1}{x(x+1)^2}$.
2. $\dfrac{2x+5}{x^3-2x^2+x}$.
3. $\dfrac{x^2+6x-1}{(x-3)^2(x-1)}$.
4. $\dfrac{8x+2}{x-x^3}$.
5. $\dfrac{3x^2-6}{(x^2-x)^2}$.
6. $\dfrac{x+2}{x^3+x^2-x-1}$.
7. $\dfrac{4+2x-x^2}{(1+x)(1-4x)^2}$.
8. $\dfrac{4+7x}{(2+3x)(1+x)^2}$.
9. $\dfrac{1-x+2x^2}{(1-x)^3}$.
10. $\dfrac{x^3-6x^2+8}{(x-1)^4}$.

CASE III

131. Some of the factors not linear.

If, when the denominator has been factored into the simplest factors which have rational coefficients, any of the factors are of higher degree than the first, the numerators of the partial fractions which have such factors for their denominators must be polynomials of degree one less than the degree of the denominator.

For example, if the factor is a quadratic, x^2+ax+b, the corresponding partial fraction is of the form $\dfrac{Ax+B}{x^2+ax+b}$. This assumption is justified, since, if the irrational factors of the denominator are $(x-m)(x-n)$, the partial fractions would be $\dfrac{M}{x-m}+\dfrac{N}{x-n}$, whose sum is in the form given above.

If any factors of higher degree than the first are equal, they must be treated just as in Case II.

Ex. 1. Separate into partial fractions $\dfrac{2x+1}{(x-1)(x^2+1)}$.

Let $$\frac{2x+1}{(x-1)(x^2+1)} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+1}.$$

Then $$\begin{aligned} 2x+1 &\equiv Ax^2 + A + Bx^2 + (C-B)x - C \\ &\equiv (A+B)x^2 + (C-B)x + (A-C). \end{aligned}$$

Equating coefficients, $$\begin{aligned} A+B&=0, \\ C-B&=2, \\ A-C&=1. \end{aligned}$$

Solving, $$A=\tfrac{3}{2},\ B=-\tfrac{3}{2},\ C=\tfrac{1}{2}.$$

Hence $$\frac{2x+1}{(x-1)(x^2+1)} = \frac{3}{2(x-1)} + \frac{1-3x}{2(x^2+1)}.$$

Ex. 2. Separate into partial fractions $\dfrac{x^3+2}{(x-2)^2(x^2+1)}$.

Let $$\frac{x^3+2}{(x-2)^2(x^2+1)} \equiv \frac{A}{(x-2)^2} + \frac{B}{x-2} + \frac{Cx+D}{x^2+1}.$$

Then $$\begin{aligned} x^3+2 &\equiv A(x^2+1) + B(x-2)(x^2+1) \\ &\qquad + (Cx+D)(x-2)^2 \qquad (1) \\ &\equiv (B+C)x^3 + (A-2B-4C+D)x^2 \\ &\qquad + (B+4C+4D)x + A-2B+4D. \end{aligned}$$

Equating coefficients of like powers of x,

$$\begin{aligned} B+C&=1, \\ A-2B-4C+D&=0, \\ B+4C-4D&=0, \\ A-2B+4D&=2. \end{aligned}$$

Since equation (1) is an identity, it is true when $x=2$. Substituting this value, we have $A=2$. Substituting this value of A in the four equations just found, they are easily solved, $$B=\tfrac{4}{5},\ C=\tfrac{1}{5},\ D=\tfrac{2}{5}.$$

Hence $$\frac{x^3+2}{(x-2)^2(x^2+1)} = \frac{2}{(x-2)^2} + \frac{4}{5(x-2)} + \frac{x+2}{5(x^2+1)}.$$

EXERCISE XCI

Separate each of the following fractions into its partial fractions:

1. $\dfrac{x^2-3}{(x+2)(x^2+1)}$.

2. $\dfrac{x^2-x+1}{(x^2+1)(x-1)^2}$.

3. $\dfrac{x^2+x}{(x-1)^2(x^2+4)}$.

4. $\dfrac{6x^2+8x}{(x^2+1)(x+5)}$.

5. $\dfrac{x^2-6x+3}{(x-1)(x^2-5x+6)}$.

6. $\dfrac{6x^3-x}{(x-1)^3}$.

7. $\dfrac{3+2x-x^2}{(1+x^2)(1-4x^2)}$.

8. $\dfrac{1}{(x^2+1)(x^2+x+1)}$.

9. $\dfrac{x^2}{x^4+x^2-2}$.

10. $\dfrac{1}{(x-1)^2(x^2+1)^2}$.

11. $\dfrac{3x-1}{x(x-2)(x-1)}$.

12. $\dfrac{1}{x^3-1}$.

13. $\dfrac{9x-10}{x^2(2x^2-2x+5)}$.

14. $\dfrac{x^5-5x-3}{(x^2+x)^2}$.

15. $\dfrac{3x+2}{(x^2-3x+3)^2}$.

16. $\dfrac{x^2-3x+2}{(x^2+2)(x^2+1)}$.

17. $\dfrac{2x^2-3x+4}{(x-3)^3}$.

18. $\dfrac{2x-5}{(x+3)(x+1)^2}$.

19. $\dfrac{5x^2-1}{(x^2+3)(x^2-2x+5)}$.

20. $\dfrac{x^2+3x+1}{x^4+x^2+1}$.

CHAPTER XVII

DETERMINANTS

132. The solution of the pair of simultaneous equations

$$a_1x + b_1y = c_1,$$
$$a_2x + b_2y = c_2,$$

is easily found, by the ordinary methods of elimination, to be

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \text{ and } y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

Each of the three quantities in these results is in the same form, of which $a_1b_2 - a_2b_1$ is typical. It has been found convenient to represent such expressions by the symbol $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv a_1b_2 - a_2b_1$. This symbol is called a **determinant** of the second order. In this notation the solution given above may be written

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \text{ and } y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

It is seen that the denominators of these fractions are the same and that they are the determinant formed by the coefficients of x and y in the given equations; also that the numerator in the value of x may be formed from this determinant by replacing a_1 and a_2 by c_1 and c_2; and that the numerator in the value of y may be formed by replacing b_1 and b_2 by c_1 and c_2.

These formulas may be used to solve any pair of simultaneous equations in two variables, and it will be shown later that similar formulas may be obtained for the solution of any set of simultaneous equations.

Ex. Solve the simultaneous equations

$$2x + 3y = 9,$$
$$3x - \quad y = 8.$$

Solution. $$x = \frac{\begin{vmatrix} 9 & 3 \\ 8 & -1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix}} = \frac{-9-24}{-2-9} = 3,$$

$$y = \frac{\begin{vmatrix} 2 & 9 \\ 3 & 8 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix}} = \frac{16-27}{-2-9} = 1.$$

EXERCISE XCII

Solve the following simultaneous equations by the method just described:

1. $8x - y = 34,$
$x + 8y = 53.$

2. $19x + 17y = 0,$
$2x - y = 53.$

3. $lx + my = n,$
$px + qy = r.$

4. $ax + by = 2ab,$
$bx - ay = b^2 - a^2.$

5. $(b+c)x + (b-c)y = 2ab,$
$(c+a)x + (c-a)y = 2ac.$

6. $bx + ay = 2ab,$
$a^2x + b^2y = a^3 + b^3.$

7. $\frac{x}{a} - \frac{y}{b} = 1,$
$\frac{x}{b} + \frac{y}{a} = \frac{a}{b}.$

133. In the determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv (a_1b_2 - a_2b_1)$ the horizontal lines, as $a_1\ b_1$, are called **rows**; the vertical lines, as $\begin{matrix} a_1 \\ a_2 \end{matrix}$, are called **columns**. The value of the determinant is seen to be unchanged, if the rows and columns are interchanged; since, by the definition of the determinant,

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

But if two rows or columns are interchanged, as

$$\begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix} = a_2b_1 - a_1b_2,$$

the sign of the determinant is seen to be changed.

If two rows or two columns are identical, the value of the determinant is zero; since

$$\begin{vmatrix} a_1 & a_1 \\ a_2 & a_2 \end{vmatrix} = a_1a_2 - a_1a_2 = 0.$$

If each element of either row or column is multiplied by any number, the value of the determinant is multiplied by that number, since

$$\begin{vmatrix} ka_1 & b_1 \\ ka_2 & b_2 \end{vmatrix} = ka_1b_2 - ka_2b_1 = k(a_1b_2 - a_2b_1) = k\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

If, therefore, any row or column contains any common factor, this may be divided out and placed before the determinant as a coefficient.

EXERCISE XCIII

Show that

1. $\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$

2. $\begin{vmatrix} a_1 & b_1 \\ a_1 & b_1 \end{vmatrix} = 0.$

3. $\begin{vmatrix} ka_1 & kb_1 \\ a_2 & b_2 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$

4. $\begin{vmatrix} a_1 + b_1 & c_1 \\ a_2 + b_2 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$

5. $\begin{vmatrix} a_1 + b_1 & c_1 + b_2 \\ a_2 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + \begin{vmatrix} b_1 & b_2 \\ a_2 & c_2 \end{vmatrix}.$

134. Similarly, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is called a **determinant of the third order.** If through any element, as a_1, lines are drawn, erasing both the row and column in which that element stands, as $\begin{vmatrix} \cancel{a_1} & \cancel{b_1} & \cancel{c_1} \\ \cancel{a_2} & b_2 & c_2 \\ \cancel{a_3} & b_3 & c_3 \end{vmatrix}$, the remaining two-rowed determinant, $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, is called the **minor** of the element a_1, and is represented by A_1. Similarly the minor of a_2, or $\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$, is represented by A_2; the minor of b_1, or $\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$, by B_1, etc.

The value of the determinant is defined as

$$a_1A_1 - a_2A_2 + a_3A_3, \text{ or}$$

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \text{ or}$$

$$a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1.$$

Let the student show by actual multiplication that any one of the following forms gives the same result and that, therefore, any one of these forms may be used as the value of the determinant:

$$-b_1B_1 + b_2B_2 - b_3B_3,$$

$$c_1C_1 - c_2C_2 + c_3C_3,$$

$$a_1A_1 - b_1B_1 + c_1C_1,$$

$$-a_2A_2 + b_2B_2 - c_2C_2,$$

$$a_3A_3 - b_3B_3 + c_3C_3.$$

The determinant may then be expanded in six ways, using the elements of any column or row, multiplied by their corresponding minors, the proper sign being given to each element as indicated in the following scheme.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

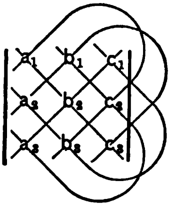
For example, we may expand the determinant $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 0 & 7 \\ 6 & 1 & 3 \end{vmatrix}$ by using the second column thus:

$$-2\begin{vmatrix} 4 & 7 \\ 6 & 3 \end{vmatrix} + 0\begin{vmatrix} 1 & 3 \\ 6 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 4 & 7 \end{vmatrix} = 60 + 5 = 65.$$

In the expanded form,

$$a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1,$$

the determinant contains six terms, each of which is composed of three elements, no two of which are taken from the same row or column. An easy method of writing out a three-rowed determinant is the following. In the accompanying scheme, form the product of the three elements lying on each line; attach the positive sign to the three terms formed by the elements on the lines drawn toward the right; attach the negative sign to the three remaining terms and take their algebraic sum. For example,



$$\begin{vmatrix} 3 & 5 & -2 \\ 1 & -4 & 8 \\ -7 & 1 & 3 \end{vmatrix} = (3)(-4)(3) + (5)(8)(-7) + (-2)(1)(1)$$
$$-(3)(8)(1) - (5)(1)(3)$$
$$-(-2)(-4)(-7)$$
$$= -36 - 280 - 2 - 24 - 15 + 56 = -301.$$

EXERCISE XCIV

Find the value of the following determinants:

1. $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix}$.

3. $\begin{vmatrix} 1 & -4 & 2 \\ 5 & 3 & 9 \\ -6 & 8 & -7 \end{vmatrix}$.

5. $\begin{vmatrix} 13 & 3 & 23 \\ 30 & 7 & 53 \\ 39 & 9 & 70 \end{vmatrix}$.

2. $\begin{vmatrix} 6 & 5 & 5 \\ 5 & 6 & 5 \\ 5 & 5 & 6 \end{vmatrix}$.

4. $\begin{vmatrix} 11 & 12 & 5 \\ -9 & 4 & -3 \\ 2 & -7 & 8 \end{vmatrix}$.

6. $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$.

7. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$

9. $\begin{vmatrix} m & x & y \\ m^2 & x^2 & y^2 \\ mn & x^3 & y^3 \end{vmatrix}.$

8. $\begin{vmatrix} xz & y & y^2 \\ yz & x & x^2 \\ xy & z & z^2 \end{vmatrix}.$

10. $\begin{vmatrix} a & x^3 & y^3 \\ o & x^2 & ay \\ a^4 & ax & y^2 \end{vmatrix}.$

135. Theorems similar to those given in § 133 for the two-rowed determinant may now be proved for the three-rowed determinant.

I. *Interchanging the columns and rows does not alter the value of the determinant.*

This follows at once from § 134, since either of the two determinants $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ may be written in the form $a_1A_1 - a_2A_2 + a_3A_3$, and are therefore equal.

II. *If two rows or columns of a determinant are interchanged, the sign of the determinant is changed, but the numerical value remains the same.*

Let $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \Delta.$ We wish to show that

$$\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} = \begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix} = -\Delta.$$

By § 134, $\Delta = a_1A_1 - b_1B_1 + c_1C_1$. By the same article,

$$\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} = b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} - a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} b_2 & a_2 \\ b_3 & a_3 \end{vmatrix}$$

$$= b_1B_1 - a_1A_1 + c_1(-C_1) = -\Delta.$$

Also

$$\begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix} = c_1\begin{vmatrix} b_2 & a_2 \\ b_3 & a_3 \end{vmatrix} - b_1\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + a_1\begin{vmatrix} c_2 & b_2 \\ c_3 & b_3 \end{vmatrix}$$

$$= c_1(-C_1) - b_1(-B_1) + a_1(-A_1) = -\Delta.$$

III. *If two rows or two columns of a determinant are identical, the value of the determinant is zero.*

If two rows or two columns of a determinant are identical, the value of the determinant is not changed by interchanging them; but we have seen that interchanging two rows or two columns changes the sign of the value of the determinant. Hence the value of the determinant is zero, since zero is the only number whose value is not changed by changing its sign.

IV. *If each element of any row or column of a determinant is multiplied by the same number, the value of the determinant is multiplied by that number.*

For $\begin{vmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{vmatrix} = ka_1A_1 - ka_2A_2 + ka_3A_3 =$

$$k(a_1A_1 - a_2A_2 + a_3A_3) = k\,\Delta.$$

Hence, if any row or column contains a common factor, that factor may be divided out and placed before the determinant as a coefficient.

V. *If the minors of any row or column are multiplied by the corresponding elements of any other row or column, with their proper signs attached, the sum of their products is zero.*

We wish to show that $a_1B_1 - a_2B_2 + a_3B_3$, or

$$a_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \text{ is zero.}$$

But this is just the development of the determinant

$$\begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix},$$

which evidently vanishes by Theorem III, since the first two columns are identical.

VI. *If equal multiples of the elements of any row or column are added to the corresponding elements of any other row or column, the value of the determinant is not changed.*

$$\text{For } \begin{vmatrix} a_1+kb_1 & b_1 & c_1 \\ a_2+kb_2 & b_2 & c_2 \\ a_3+kb_3 & b_3 & c_3 \end{vmatrix} = (a_1+kb_1)A_1 - (a_2+kb_2)A_2 + (a_3+kb_3)A_3$$
$$= a_1A_1 - a_2A_2 + a_3A_3 + k(b_1A_1 - b_2A_2 + b_3A_3) = \Delta,$$

since, by V, $b_1A_1 - b_2A_2 + b_3A_3 = 0$.

These theorems will be found useful in obtaining the value of any determinant.

Ex. 1. Find the value of the determinant $\begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 3 & 5 & 8 \end{vmatrix}$.

$$\begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 3 & 5 & 8 \end{vmatrix} = \begin{vmatrix} -10 & 0 & -13 \\ 4 & 1 & 6 \\ -17 & 0 & -22 \end{vmatrix} = \begin{vmatrix} 10 & 13 \\ 17 & 22 \end{vmatrix}$$

$$= \begin{vmatrix} 10 & 13 \\ 7 & 9 \end{vmatrix} = 90 - 91 = -1.$$

The second determinant is obtained by multiplying the second row of the given determinant by -3 and adding it to the first; also the second row by -5 and adding it to the third. Then, if we expand the determinant by the second column, it reduces to $\begin{vmatrix} -10 & -13 \\ -17 & -22 \end{vmatrix}$. The factor -1 may now be divided out of each row, and the value of the determinant is seen to be $10 \times 22 - 13 \times 17$; or we may reduce it to smaller numbers by subtracting the first row from the second, as shown above.

The central idea of this method is to use Theorem VI in such a way that all the elements of some row or column become zero except one. Then, if we expand by that row or column, the determinant reduces to a two-rowed determinant. This method is, perhaps, no easier for the three-rowed determinant than the one described in § 134; but since it is by far the easier method in evaluating determinants of higher orders, the student should become familiar with it here in the simple case.

It is important to notice that the row or column which is added to the others must itself remain unchanged in the new determinant.

Ex. 2. Find the value of the determinant $\begin{vmatrix} 3 & -4 & 2 \\ -5 & 8 & 7 \\ 4 & 3 & 6 \end{vmatrix}$.

$$\begin{vmatrix} 3 & -4 & 2 \\ -5 & 8 & 7 \\ 4 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 7 & -4 & 10 \\ -13 & 8 & -9 \\ 1 & 3 & 0 \end{vmatrix} = \begin{vmatrix} 7 & -25 & 10 \\ -13 & 47 & -9 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} -25 & 10 \\ 47 & -9 \end{vmatrix} = 225 - 470 = -245.$$

Subtract the second column from the first; also multiply the second column by 2 and subtract it from the third. In the second determinant multiply the first column by 3 and subtract it from the second; then expand by the third row.

EXERCISE XCV

Find the value of the following determinants:

1. $\begin{vmatrix} 13 & 3 & 23 \\ 30 & 7 & 53 \\ 39 & 9 & 70 \end{vmatrix}$.

2. $\begin{vmatrix} 15 & 17 & 16 \\ 12 & 18 & 14 \\ 19 & 17 & 13 \end{vmatrix}$.

3. $\begin{vmatrix} 3 & -1 & 2 \\ 2 & 1 & -3 \\ -4 & 2 & 5 \end{vmatrix}$.

4. $\begin{vmatrix} -4 & 8 & 15 \\ 6 & -5 & -4 \\ 14 & 11 & 9 \end{vmatrix}$.

5. $\begin{vmatrix} 19 & -12 & 7 \\ 9 & 18 & -21 \\ -11 & -24 & 49 \end{vmatrix}$.

6. $\begin{vmatrix} 3 & 2 & 7 \\ 4 & 1 & 5 \\ 6 & 8 & 9 \end{vmatrix}$.

7. $\begin{vmatrix} 3 & 4 & 2 \\ 15 & -6 & -7 \\ 21 & 12 & 1 \end{vmatrix}$.

8. $\begin{vmatrix} -42 & 9 & -3 \\ -14 & -63 & 21 \\ 70 & 45 & -10 \end{vmatrix}$.

9. $\begin{vmatrix} 1\frac{1}{2} & -1\frac{2}{3} & -1\frac{3}{4} \\ -1\frac{4}{5} & 1\frac{5}{6} & 1\frac{7}{8} \\ 1\frac{1}{5} & -1\frac{1}{3} & -1\frac{1}{2} \end{vmatrix}$.

10. $\begin{vmatrix} 7 & 14 & 5 \\ \frac{1}{4} & 2 & 10 \\ \frac{1}{5} & 4 & 1\frac{2}{7} \end{vmatrix}$.

136. Solution of simple simultaneous equations in three variables.

Consider the general set of simultaneous equations in three variables.

$$(1)\quad a_1x+b_1y+c_1z=h_1,$$
$$(2)\quad a_2x+b_2y+c_2z=h_2,$$
$$(3)\quad a_3x+b_3y+c_3z=h_3.$$

Let the determinant of the coefficients $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_2 & c_3 \end{vmatrix}$ be represented by Δ.

To solve these equations for x, multiply (1) by A_1, the minor of a_1 in Δ; (2) by $-A_2$, the minor a_2; and (3) by A_3 and add. We have

$$(a_1A_1-a_2A_2+a_3A_3)x+(b_1A_1-b_2A_2+b_3A_3)y+$$
$$(c_1A_1-c_2A_2+c_3A_3)z=h_1A_1-h_2A_2+h_3A_3.$$

But by V, § 135, the coefficients of y and z are zero; also, by the definition of a determinant, the coefficient of x is Δ, and the second member is the determinant

$$\begin{vmatrix} h_1 & b_1 & c_1 \\ h_2 & b_2 & c_2 \\ h_3 & b_3 & c_3 \end{vmatrix}.$$

Then

$$x=\frac{\begin{vmatrix} h_1 & b_1 & c_1 \\ h_2 & b_2 & c_2 \\ h_3 & b_3 & c_3 \end{vmatrix}}{\Delta}.$$

If we multiply (1) by $-B_1$, (2) by B_2, and (3) by $-B_3$, and add, we have

$$-(a_1B_1-a_2B_2+a_3B_3)x+(-b_1B_1+b_2B_2-b_3B_3)y-$$
$$(c_1B_1-c_2B_2+c_3B_3)z=-h_1B_1+h_2B_2-h_3B_3.$$

Hence,
$$y=\frac{\begin{vmatrix} a_1 & h_1 & c_1 \\ a_2 & h_2 & c_2 \\ a_3 & h_3 & c_3 \end{vmatrix}}{\Delta}.$$

In like manner,
$$z=\frac{\begin{vmatrix} a_1 & b_1 & h_1 \\ a_2 & b_2 & h_2 \\ a_3 & b_3 & h_3 \end{vmatrix}}{\Delta}.$$

Hence the solution of three simultaneous simple equations may be written as fractions, in each of which the denominator is Δ*; the numerator of the value of* x *is obtained from* Δ *by replacing the coefficients of* x *by the constants in the second member; the numerator of the value of* y*, by replacing the coefficients of* y *by the constants in the second member; and the numerator of the value of* z*, by replacing the coefficients of* z *by the constants in the second member.*

Ex. Solve the simultaneous equations,

$$\begin{aligned} x-2y+3z&=2,\\ 2x-3y+\ \ z&=1,\\ 3x-\ \ y+2z&=9. \end{aligned}$$

$$x=\frac{\begin{vmatrix} 2 & -2 & 3 \\ 1 & -3 & 1 \\ 9 & -1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & -2 & 3 \\ 2 & -3 & 1 \\ 3 & -1 & 2 \end{vmatrix}}=\frac{54}{18}=3.$$

$$y=\frac{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 9 & 2 \end{vmatrix}}{18}=\frac{36}{18}=2. \qquad z=\frac{\begin{vmatrix} 1 & -2 & 2 \\ 2 & -3 & 1 \\ 3 & -1 & 9 \end{vmatrix}}{18}=\frac{18}{18}=1.$$

EXERCISE XCVI

Solve the following systems of simultaneous equations:

1. $2x+3y+4z=20,$
$3x+4y+5z=26,$
$3x+5y+6z=31.$

2. $3x-2y+z=2,$
$2x+3y-z=5,$
$x+y+z=6.$

3. $2x+y+z=16,$
$x+2y+z=9,$
$x+y+2z=3.$

4. $4x-3y+2z=21,$
$5x+9y-7z=-66,$
$9x+8y-3z=-49.$

5. $6x+2y-5z=13,$
$3x+3y-2z=13,$
$7x+5y-3z=26.$

6. $5x+2y=14,$
$y-6z=-15,$
$x+2y+z=0.$

7. $2x-5y=-19,$
$3y+4z=-7,$
$2z-5x=2.$

8. $y+z-x=2a,$
$z+x-y=2b,$
$x+y-z=2c.$

9. $x-ay+a^2z=a^3,$
$x-by+b^2z=b^3,$
$x-cy+c^2z=c^3.$

10. $ax+cy+bz=a^2+2bc,$
$cx+by+az=b^2+2ac,$
$bx+ay+cz=c^2+2ab.$

137. Determinants of higher orders.

A determinant of the fourth order consists of four rows and four columns. Its value is defined as follows:

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1A_1 - a_2A_2 + a_3A_3 - a_4A_4,$$

where A_i, the minor of a_i, is the three-rowed determinant which remains when the row and column in which a_i stands are erased. Similar definitions may be given for determinants of the fifth and higher orders.

All of the definitions of § 134 and the theorems of § 135, which are there stated for determinants of the third order, may be applied to determinants of any order. Also the general method for solving simultaneous equations in three variables, given at the end of § 136, may be extended to the solution of simultaneous equations in any number of variables.

The method of finding the value of a determinant of the fourth order is illustrated in the following problem:

Ex. Find the value of $\begin{vmatrix} 2 & 4 & 6 & 5 \\ -3 & 1 & 2 & 7 \\ 5 & -3 & 4 & 2 \\ 1 & 2 & -8 & -9 \end{vmatrix}$.

Multiply the second row by -4 and add to the first; by 3 and add to the third; by -2 and add to the fourth.

$$\begin{vmatrix} 14 & 0 & -2 & -23 \\ -3 & 1 & 2 & 7 \\ -4 & 0 & 10 & 23 \\ 7 & 0 & -12 & -23 \end{vmatrix} = 46\begin{vmatrix} 14 & -1 & -1 \\ -4 & 5 & 1 \\ 7 & -6 & -1 \end{vmatrix} = 46\begin{vmatrix} 14 & -1 & -1 \\ 10 & 4 & 0 \\ -7 & -5 & 0 \end{vmatrix}$$

$$= 46\begin{vmatrix} 10 & 4 \\ 7 & 5 \end{vmatrix} = 46 \times 22 = 1012.$$

The three-rowed determinant is here reduced by adding the first row to the second, and subtracting it from the third.

EXERCISE XCVII

Determine the value of the following determinants:

1. $\begin{vmatrix} 1 & 1 & 1 & 4 \\ 2 & 4 & 1 & 8 \\ 4 & 1 & 2 & 13 \\ 2 & 4 & 2 & 11 \end{vmatrix}.$

2. $\begin{vmatrix} 7 & -1 & 0 & 5 \\ -2 & 3 & -2 & 2 \\ 0 & -1 & 5 & 3 \\ 5 & 1 & 3 & 4 \end{vmatrix}.$

3. $\begin{vmatrix} 1 & 4 & 3 & 1 \\ 2 & 8 & 2 & 5 \\ 6 & 4 & 1 & 2 \\ 2 & 5 & 3 & 3 \end{vmatrix}.$

4. $\begin{vmatrix} 13 & 0 & 3 & 14 \\ 3 & -2 & 1 & 4 \\ -2 & 5 & 2 & 9 \\ 18 & -3 & 5 & 22 \end{vmatrix}.$

5. $\begin{vmatrix} 5 & -10 & 11 & 0 \\ -10 & -11 & 12 & 4 \\ 11 & 12 & -11 & 2 \\ 0 & 4 & 2 & -6 \end{vmatrix}.$

6. $\begin{vmatrix} 2 & 3 & 4 & 5 \\ 7 & 8 & 9 & 11 \\ 17 & 7 & 13 & 2 \\ 5 & 4 & 1 & 6 \end{vmatrix}.$

7. $\begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix}.$

8. $\begin{vmatrix} 0 & a & b & c \\ a & 0 & c & b \\ b & c & 0 & a \\ c & b & a & 0 \end{vmatrix}.$

9. $\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & a & b \\ 1 & a & 0 & c \\ 1 & b & c & 0 \end{vmatrix}.$

10. $\begin{vmatrix} 0 & a & b & c \\ a & 0 & x & y \\ b & x & 0 & z \\ c & y & z & 0 \end{vmatrix}.$

11. $\begin{vmatrix} a & b & c & d \\ -a & b & c & d \\ -a & -b & c & d \\ -a & -b & -c & d \end{vmatrix}.$

12. $\begin{vmatrix} a & a & b & a \\ b & b & b & a \\ b & a & a & a \\ b & a & b & b \end{vmatrix}.$

Solve the following systems of simultaneous equations:

13. $3u - 2y = 2,$
$2x + 3y = 39,$
$5x - 7z = 11,$
$4y + 3z = 41.$

14. $x + 2y = 9,$
$3y + 4z = 14,$
$7z + u = 5,$
$2u + 5x = 8.$

15. $x + y + z = 2,$
$x + y + u = 3,$
$x + z + u = 4,$
$y + z + u = 0.$

16. $x + y + z - u = 11,$
$x + y - z + u = 17,$
$x - y + z + u = 9,$
$-x + y + z + u = 12.$

17. $2x + 3y - z + 4u = 41,$
$x - 5y + 4z - u = 6,$
$3x - y - z + u = 22,$
$4x + y + z - 5u = 36.$

18. $2x - 4y + 3z + 4u = -3,$
$x - 10y - 3z - 7u = 2,$
$5x + 8y + 9z + 3u = 9,$
$3x - 2y + 6z + 5u = -1.$

138. Simultaneous equations.

In the solution of the set of equations given in § 136 the denominator of each unknown was found to be the determinant formed by the coefficients of the unknowns. In case this determinant does not vanish, a definite solution was obtained, if the number of equations in the set was equal to the number of unknowns. If this determinant does vanish, the equations are either dependent, with an indefinite number of solutions, or contradictory and without any solution.

If there are more equations than variables, they will not in general be simultaneous, unless some relation exists between the coefficients.

Consider the three equations in x and y,

$$\begin{aligned} a_1x + b_1y + c_1 &= 0, \\ a_2x + b_2y + c_2 &= 0, \\ a_3x + b_3y + c_3 &= 0. \end{aligned}$$

The solution of the first two is

$$x = \frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{-\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

If the three equations are simultaneous, these values must satisfy the third equation, or

$$a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_3\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0, \text{ or } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

This is then the condition which must be satisfied if the three equations are simultaneous. It is called the **eliminant** of the set of equations, and a similar condition may be obtained in any case where the number of equations is one more than that of the variables.

139. Homogeneous equations.

If a simple equation in any number of variables lacks the constant term, it is said to be homogeneous. If such an equation is satisfied by certain values of the unknowns, it will also be satisfied by any multiples of these values. It is not, therefore, possible to solve a set of such equations in definite values; but the ratios between the unknowns may be determined in case the number of equations is one less than the number of the unknowns. If the number of equations equals the number of unknowns, the only value which will satisfy the equations is zero, unless a relation exists between the coefficients.

Consider the set of three homogeneous equations,

$$\begin{aligned} a_1x + b_1y + c_1z &= 0, \\ a_2x + b_2y + c_2z &= 0, \\ a_3x + b_3y + c_3z &= 0. \end{aligned}$$

If we suppose z not to be zero, these are equivalent to

$$\begin{aligned} a_1\frac{x}{z} + b_1\frac{y}{z} + c_1 &= 0, \\ a_2\frac{x}{z} + b_2\frac{y}{z} + c_2 &= 0, \\ a_3\frac{x}{z} + b_3\frac{y}{z} + c_3 &= 0. \end{aligned}$$

Any pair of these equations may be solved for the ratios $\frac{x}{z}$ and $\frac{y}{z}$, but the three are simultaneous only when

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

This is again spoken of as the eliminant of the set of equations, and similar theorems may be obtained for sets of homogeneous equations in any number of variables.

EXERCISE XCVIII

Determine whether each of the following sets of equations can be solved, and if not, determine the reason.

1. $$x + y - z = 8,\quad 2x - 3y + 2z = 10,\quad x + 11y - 9z = 20.$$

2. $$x + y - z = 8,\quad 2x - 3y + 2z = 10,\quad 3x - 2y + z = 15.$$

3. $$x + y - z = 8,\quad 2x - 3y + 2z = 10,\quad 3x - 2y - z = 15.$$

Determine the ratios of the unknowns in each of the following sets of equations:

4. $$3x - 4y - 2z = 0,\quad 7x - 9y - 7z = 0.$$

5. $$2x - 3y + 4z = 0,\quad x + 2y - 5z = 0.$$

6. Show that the condition which must be satisfied if the three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) lie on the graph of an equation of the form

$$ax + by + c = 0 \text{ is } \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

7. Show that the condition which must be satisfied if the graphs of the three equations,

$$a_1x + b_1y + c_1 = 0,\quad a_2x + b_2y + c_2 = 0,\quad a_3x + b_3y + c_3 = 0,$$

pass through the same point (x_1, y_1) is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

8. Determine the equation in x, y, and z, which is satisfied by the three sets of values (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3).

CHAPTER XVIII

THEORY OF EQUATIONS

140. Roots of equations. The term **roots** of an equation of any degree in a single variable is used to designate those constants which reduce the equation to an identity, when they are substituted for the variable throughout the equation.

In Chapter VI the methods of finding the roots of any equation of the second degree were described and a solution of the general quadratic, $ax^2 + bx + c = 0$, was obtained. Solutions may also be obtained for the general equations of the third and fourth degree. But for the general equations of the fifth and higher degrees no such solutions are possible. We may, however, obtain the roots of any given numerical equation, exactly if these roots are rational, or approximately if the roots are irrational. It is the purpose of this chapter to discuss the methods by which these roots may be obtained.

141. Let $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$ be any equation of degree n, in which the coefficients a_0, a_1, a_2, $\cdots a_n$ are either positive or negative numbers or zero, except that a_0 shall always be positive. The first member of this equation will be represented by the abbreviation $f(x)$, or, if it is desirable to indicate its degree, by $f_n(x)$. The symbol $f(c)$ is used to represent the value of the $f(x)$, in which the constant c has been substituted throughout for the variable x. If $f(c) = 0$, then c is a root of the equation $f(x) = 0$.

THE REMAINDER THEOREM

142. THEOREM I. *When $f(x)$ is divided by $x-c$, the remainder is $f(c)$, or the value of $f(x)$ when c is substituted for x.*

Let $f(x)$ be divided by $x-c$, and let Q represent the quotient and R the remainder. Then

$$f(x) = Q\,(x-c) + R.$$

Since the remainder must be of lower degree than the divisor, R does not contain x and is, therefore, constant. Since the equality holds for all values of x, we may substitute $x = c$. Hence

$$f(c) = Q\,(c-c) + R, \text{ or}$$
$$f(c) = R.$$

EXERCISE XCIX

1. If $f(x) = x^3 - 6x + 4$, find $f(0), f(3), f(-2)$.

2. If $f(x) = x^3 - 3x^2 + 5x - 8$, find $f(2)$ by substitution, and show that it is equal to the remainder obtained in dividing $f(x)$ by $x-2$.

3. If $f(x) = x^4 - x^2 + 3x - 8$, find $f(5)$ by substitution and show that it is equal to the remainder obtained in dividing $f(x)$ by $x-5$.

4. Divide $ax^3 + bx^2 + cx + d$ by $x - h$ and show that the remainder is $f(h)$, or $ah^3 + bh^2 + ch + d$.

THE FACTOR THEOREM

143. THEOREM II. *If a is a root of the equation $f(x) = 0$, then $x - a$ is a factor of $f(x)$; and conversely, if $x - a$ is a divisor of $f(x)$, a is a root of the equation $f(x) = 0$.*

By the theorem of the last article, the remainder R obtained after dividing $f(x)$ by $x-\alpha$ is $f(\alpha)$, or $R=f(\alpha)$. If α is a root of the equation $f(x)=0, f(\alpha)=0$, and $R=0$. Hence $x-\alpha$ is a factor of $f(x)$.

Conversely, if $R=0$, $f(\alpha)=0$, and α is a root of the equation.

NUMBER OF ROOTS

144. It is here assumed that every equation $f(x)=0$ has at least one root; or, in other words, that there is some number, rational, irrational, or imaginary, which will satisfy the equation when it is substituted for x. The proof of this statement is too difficult to be presented here, and it is therefore assumed without proof.

With this assumption we shall now prove the following theorem.

THEOREM III. *Every equation of degree n has n roots.*

Let $f_n(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n=0.$

According to our assumption this equation must have a root. Let α_1 represent this root. Then, by § 143, $f_n(x)$ is divisible by $x-\alpha_1$ and $f_n(x)=(x-\alpha)f_{n-1}(x)$, where $f_{n-1}(x)$ is of degree $n-1$. Also $f_{n-1}(x)=0$ must, by our assumption, have a root, which we may call α_2. Then $f_{n-1}(x)$ is divisible by $x-\alpha_2$, and $f_n(x)=(x-\alpha_1)(x-\alpha_2)$ $f_{n-2}(x)$. Proceed in this way until the quotient is of the first degree. The coefficient of x in this quotient is a_0. Dividing by a_0, we have

$$f_n(x)=a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n).$$

There are evidently just n of these factors, and since the equation is satisfied by every value of x which reduces one of these factors to zero, it has n roots. These are not necessarily all distinct, since any number of the quantities α_1, α_2, α_3, etc., may be equal to each other. If two of them have the same value, it is called a double root; if three of them, a triple root; if r of them, a multiple root of order r. In the statement of the theorem, "every equation has n roots," it is understood that a multiple root of order r should be counted as r roots.

THEOREM IV. *An equation of degree n cannot have more than n distinct roots.*

If the equation $f_n(x) = 0$ has more than n distinct roots, at least one of these must be different from α_1, α_2, $\cdots$ α_n. Let β be such a root. Then

$$f_n(\beta) = a_0(\beta - \alpha_1)(\beta - \alpha_2) \cdots (\beta - \alpha_n).$$

But this product cannot reduce to zero unless one of the binomial factors is zero; but this cannot be, since β has been chosen distinct from any α. Hence $f_n(\beta) \neq 0$, and β is not a root of $f_n(x) = 0$.

In the proof of this theorem it is understood that the equation is actually of degree n, or that a_0 is not zero.

From the theorems of this article it appears that the problem of solving any equation, $f(x) = 0$, is identical with that of factoring $f(x)$. Also that, if the roots of any equation are given, *the equation may be determined by multiplying together n binomial factors formed by writing x minus each of the roots and placing the product equal to zero.*

Ex. 1. Find the roots of $x^3 - 5x^2 + 6x = 0$.

The equation may be written in the form

$$x(x - 3)(x - 2) = 0.$$

Hence the roots are 0, 3, and 2.

Ex. 2. Form the equation whose roots are -2, 3, and 4.

The equation must be

$$(x+2)(x-3)(x-4)=0,$$

or

$$x^3-5x^2-2x+24=0.$$

EXERCISE C

1. Determine by division whether each of the following numbers is a root of the equation opposite it:

(*a*) 3 $x^3-2x^2-9=0,$
(*b*) 5 $x^4-3x^3+20x-15=0,$
(*c*) 2 $x^3-2x^2-5x+6=0.$

2. Find by division all the roots of

(*a*) $x^3-6x^2+3x+10=0,$
(*b*) $x^3+3x^2-28x=0.$

3. Form the equations whose roots are

(*a*) $3, -5, 6.$
(*b*) $\frac{1}{3}, \frac{1}{3}, -3.$
(*c*) $2+\sqrt{3}, 2-\sqrt{3}, \pm 2.$
(*d*) $-1, -2, -3.$
(*e*) $5, \frac{4\pm 2\sqrt{3}}{3}.$
(*f*) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, -3.$

145. Synthetic division. It was shown in §§ 142, 143, that, if α is a root of the equation $f(x)=0$, $f(x)$ is divisible by $x-\alpha$, and if α is not a root, that $R=f(\alpha)$. When the roots of an equation are rational, this theorem may be used to determine these roots, and a short method of performing the division is desirable. It will also appear that the process to be described is the easiest method of finding the value of $f(x)$, when any number is substituted for x, by using the fact that $R=f(\alpha)$. The process is illustrated in the following example:

Ex. 1. Divide $x^4+2x^3-3x^2+10x+15$ by $x-2$. Using the ordinary process of division,

$$\begin{array}{r|rrrrr|l}
\underline{x-2} & x^4 & +2x^3 & -3x^2 & +10x & +15 & \underline{x^3+4x^2+5x+20} \\
 & \underline{x^4} & \underline{-2x^3} & & & & \\
 & & 4x^3 & -3x^2 & & & \\
 & & \underline{4x^3} & \underline{-8x^2} & & & \\
 & & & 5x^2 & +10x & & \\
 & & & \underline{5x^2} & \underline{-10x} & & \\
 & & & & 20x & +15 & \\
 & & & & \underline{20x} & \underline{-40} & \\
 & & & & & 55 &
\end{array}$$

It is easily seen that in this process many unnecessary quantities are written. We may omit the various powers of x entirely, since they may be easily supplied from the position of the term. Also, it is not necessary to repeat the first term of each product, since it is always the same as the term above it. The process would then appear in the form

$$\begin{array}{r|rrrrr|l}
\underline{-2} & 1 & +2 & -3 & +10 & +15 & \underline{1+4+5+20} \\
 & & \underline{-2} & & & & \\
 & & 4 & -3 & & & \\
 & & & \underline{-8} & & & \\
 & & & 5 & +10 & & \\
 & & & & \underline{-10} & & \\
 & & & & 20 & +15 & \\
 & & & & & \underline{-40} & \\
 & & & & & 55 &
\end{array}$$

A further reduction of the labor involved is obtained by omitting to bring down the term of the dividend and moving the whole work up into three horizontal lines, thus:

$$\begin{array}{r|r} \underline{-2} & 1+2-3+10+15 \\ & -2-8-10-40 \\ \cline{2-2} & 1+4+5+20+55 \end{array}$$

Each remainder will be the next term in the quotient, and the quantity to be subtracted from the following term is therefore found by multiplying each remainder by -2. Since errors are less likely to occur in addition than in subtraction, it is customary to change the sign of the term in the divisor and add, thus:

$$\begin{array}{r|l} 1+2-3+10+15 & \underline{2} \\ +2+8+10+40 & \\ \hline 1+4+5+20+55 & \end{array}$$

This change of sign is also desirable from the fact that it is $+2$ and not -2 which we are substituting in $f(x)$.

The quotient of the division may be obtained at once by attaching x with the proper exponent to all the terms of the third row except the last, $x^3+4x^2+5x+20$, while the last term, 55, is the remainder, or $f(2)$.

Ex. 2. Divide x^4-3x^2-56 by $x+3$.

Writing the coefficients of all the powers of x and using -3 as divisor, we have

$$\begin{array}{rrrrr|l} 1 & 0 & -3 & 0 & -56 & \underline{-3} \\ & -3 & +9 & -18 & 54 & \\ \hline 1 & -3 & +6 & -18 & -2 & \end{array}$$

The quotient is $x^3-3x^2+6x-18$, and the remainder, or $f(-3)$, is -2.

We may now formulate the following rule for the synthetic division of $f(x)$ by $x-c$:

Write the coefficients a_0, a_1, *etc., of* $f(x)$ *in their proper order, inserting a zero where any power of* x *is wanting. Multiply* a_0 *by* c *and add the product to* a_1*; multiply the sum by* c *and add the product to* a_2, *etc.* a_0 *and the various sums obtained, except the last, will be the coefficients of the quotient. The last sum will be the remainder, or* $f(c)$.

INTEGRAL ROOTS

146. We are now able to solve any equation whose roots are all integers; for, as we have seen, it is only necessary to factor $f(x)$, and this can always be done by a few tests in case the roots are integers. Evidently $f(x)$ cannot be exactly divisible by $x-c$ unless c is a factor of the last term. It is therefore only necessary to try as possible roots the factors of that term. Synthetic division greatly shortens the labor of such trials.

Ex. 1. Find the integral roots of

$$x^4-15x^2+10x+24=0.$$

The only integers which may be roots are the factors of 24, or ± 1, ± 2, ± 3, ± 4, ± 6, ± 8, ± 12. By inspection $+1$ is not a root, but -1 is a root. Dividing by $x+1$, we have

$$\begin{array}{rrrrr|l} 1 & 0 & -15 & +10 & +24 & \underline{-1} \\ & -1 & +1 & +14 & -24 & \\ \cline{1-5} 1 & -1 & -14 & +24 & 0 & \end{array}$$

The resulting or depressed equation is

$$x^3-x^2-14x+24=0.$$

Testing 2,

$$\begin{array}{rrrr|l} 1 & -1 & -14 & +24 & \underline{2} \\ & +2 & +2 & -24 & \\ \cline{1-4} 1 & +1 & -12 & 0 & \end{array}$$

we find that it is a root. The depressed equation,

$$x^2 + x - 12 = 0,$$

is now a quadratic and may be solved by the ordinary process. The roots of the original equation are, therefore, -4, -1, 2, and 3.

Ex. 2. Find the integral roots of $x^3 - 14x - 15 = 0$.

The only integers which may be roots are the factors of 15, or ± 1, ± 3, ± 5. Trying $+1$, -1, $+3$, they are found not to be roots. Trying -3,

$$\begin{array}{rrrr|l} 1 & 0 & -14 & -15 & \underline{-3} \\ & -3 & +9 & +15 & \\ \hline 1 & -3 & -5 & 0 & \end{array}$$

we find that -3 is a root and that the quotient obtained by dividing $f(x)$ by $x + 3$ is $x^2 - 3x - 5$. Solving,

$$x^2 - 3x - 5 = 0,\ x = \frac{3 \pm \sqrt{29}}{2}.$$

Hence the three roots of the given equation are -3, $\frac{3 + \sqrt{29}}{2}$, and $\frac{3 - \sqrt{29}}{2}$.

EXERCISE CI

Obtain by synthetic division the quotient and remainder in each of the following problems:

1. $(x^3 - 9x^2 + 27x - 27) \div (x - 3)$.
2. $(x^3 - 5x^2 - 46x - 40) \div (x + 4)$.
3. $(2x^3 - 19x^2 + 26x - 16) \div (x - 8)$.
4. $(x^5 - 1) \div (x - 1)$.
5. $(x^5 - 4x^3 + 3x - 4) \div (x + 5)$.
6. $(x^4 - 6x^3 + x^2 - 15) \div (x - 6)$.

7. $(3x^4+6x^2-8x+13)\div(x+3)$.

8. $(6x^5-14x^3+12x^2-17)\div(x-8)$.

9. $(5x^5+x^2-1)\div(x+5)$.

10. $(3x^4-40x^3+130x^2-120x+27)\div(x-9)$.

Find the integral roots of the following equations:

11. $x^3-8x^2+13x-6=0$.

12. $x^3+2x^2-23x-60=0$.

13. $x^4-45x^2-40x+84=0$.

14. $x^4-11x^2+18x-8=0$.

15. $x^3-3x^2+x+2=0$.

16. $x^4-4x^3-8x+32=0$.

17. $x^3-5x^2-2x+24=0$.

18. $x^3+6x^2-6x-63=0$.

19. $x^4-3x^3-21x^2+43x+60=0$.

20. $x^4+2x^3-25x^2-26x+120=0$.

GRAPHS OF EQUATIONS

147. By the method of Chapter VII it is possible to construct the graphs of equations of any degree. Consider the equation

$$y=x^3-7x^2+4x+12.$$

The table of values may be constructed most easily by using synthetic division, thus:

$$\begin{array}{rrrr|l} 1 & -7 & +4 & +12 & \underline{1} \\ & 1 & -6 & -2 & \\ \hline 1 & -6 & -2 & +10 & \end{array} \qquad \begin{array}{rrrr|l} 1 & -7 & +4 & +12 & \underline{2} \\ & 2 & -10 & -12 & \\ \hline 1 & -5 & -6 & 0 & \end{array} \qquad \begin{array}{rrrr|l} 1 & -7 & +4 & +12 & \underline{3} \\ & 3 & -12 & -24 & \\ \hline 1 & -4 & -8 & -12 & \end{array}$$

$$\begin{array}{rrrr|l} 1 & -7 & +4 & +12 & \underline{4} \\ & 4 & -12 & -32 & \\ \hline 1 & -3 & -8 & -20 & \end{array} \qquad \begin{array}{rrrr|l} 1 & -7 & +4 & +12 & \underline{5} \\ & 5 & -10 & -30 & \\ \hline 1 & -2 & -6 & -18 & \end{array} \qquad \begin{array}{rrrr|l} 1 & -7 & +4 & +12 & \underline{6} \\ & 6 & -6 & -12 & \\ \hline 1 & -1 & -2 & 0 & \end{array}$$

$$\begin{array}{rrrr|l} 1 & -7 & +4 & +12 & \underline{7} \\ & 7 & 0 & +28 & \\ \hline 1 & 0 & +4 & +40 & \end{array} \qquad \begin{array}{rrrr|l} 1 & -7 & +4 & +12 & \underline{-1} \\ & -1 & +8 & -12 & \\ \hline 1 & -8 & +12 & 0 & \end{array} \qquad \begin{array}{rrrr|l} 1 & -7 & +4 & +12 & \underline{-2} \\ & -2 & +18 & -44 & \\ \hline 1 & -9 & +22 & -32 & \end{array}$$

In this way we obtain the following table of values and graph:

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
$x =$	-2	-1	0	1	2	3	4	5	6	7
$y =$	-32	0	12	10	0	-12	-20	-18	0	40

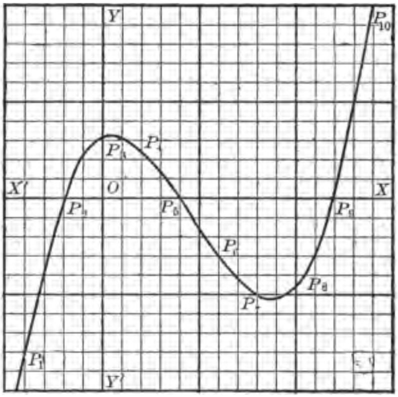


FIG. 15.

In constructing the graph, two horizontal units represent a unit in the value of x, while each vertical unit represents four units in the value of y.

At each of the points where the graph crosses the X-axis the value of y is zero and the values of x at these points are solutions of the equation

$$x^3 - 7x^2 + 4x + 12 = 0.$$

In the example these are $x = -1, 2, 6$, and the roots have been completely determined by forming the graph. This will always be the case when the roots are integers. If the roots are not integers, the graph indicates between which two integers the real roots lie, and if the graph is carefully constructed, close approximations to the correct roots may be obtained by measurement. As an illustration, consider the graph of the equation

$$y = x^3 - 3x^2 - 4x + 10.$$

Constructing a table of values and the corresponding graph, we have:

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
$x =$	-3	-2	-1	0	1	2	3	4
$y =$	-32	-2	10	10	4	-2	-2	10

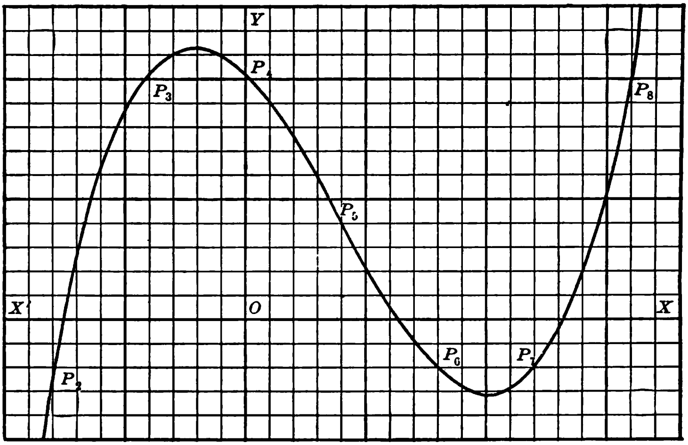


Fig. 16.

In constructing the graph four horizontal units represent a unit in the value of x, while each vertical unit represents a single unit in the value of y.

Here we see that the curve crosses the X-axis between -2 and -1, 1 and 2, 3 and 4, and measurement of the figure gives -1.9, 1.6, and 3.3 as close approximations of the roots.

148. In constructing a table of values and the corresponding points, it is convenient to be able to determine maximum and minimum values of x, beyond which it is unnecessary to proceed. In substituting positive numbers by synthetic division, if the signs of all the coefficients of the quotient and the remainder are positive, it may easily be seen that an increase in the number substituted can only increase the coefficients and the remainder. Hence beyond this point the graph continues to go upward indefinitely and it is unnecessary to determine more points in that direction. Again, in substituting negative numbers, if the signs of the coefficients and the remainder are alternately plus and minus, an increase in the absolute value of the number substituted can only increase the remainder.

In the first illustration given above, when 1, 2, 3, 4, 5, or 6 is substituted, the signs of the coefficients and the remainder vary, but when 7 is substituted, all the coefficients and the remainder are positive and it is not necessary to substitute larger positive values of x. Also when -2 is tried, the coefficients and the remainder alternate in sign, and we have reached the limit of possible roots in the negative direction.

149. Continuity. In constructing the graph in the preceding article we have assumed that it will be a smooth curve without breaks; or that, as x changes gradually from a to b, $f(x)$ changes gradually from $f(a)$ to $f(b)$, taking on all the intermediate values. Such a curve is said to be *continuous*. The graphs of all equations in the form $y = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$ are continuous, but a general proof of this fact is too difficult to be given here. It is therefore assumed without proof.

150. Location of the roots. The assumption of the preceding article enables us to prove the following theorem:

THEOREM V. *If $f(a)$ and $f(b)$ have unlike signs, there must be an odd number of real roots between a and b.*

For, if $f(x)$ changes continuously from a positive to a negative value, or the reverse, the graph of the equation $y=f(x)$ must cross the X-axis an odd number of times and there must be an odd number of roots between a and b.

Let the student prove in a similar manner the following theorem:

THEOREM VI. *If $f(a)$ and $f(b)$ have like signs, there must be no roots or an even number of roots between a and b.*

These theorems are of great importance in solving numerical equations, for by their aid we are usually enabled to locate the real roots between two integers.

For example, in the second equation of § 147, $x^3-3x^2-4x+10=0$, it was found that $f(-2)=-2$ and $f(0)=10$. Hence there must be at least one root between -2 and 0. Again, $f(1)=4$ and $f(4)=10$. Since these are both positive, there must be either no root or, as we found, two roots between 1 and 4.

EXERCISE CII

Construct carefully the graph of each of the following equations and determine by measurement the approximate value of each of the real roots:

1. $x^2-6x+10=0$.
2. $x^3+12x^2+50x+48=0$.
3. $x^3+12x^2+44x+48=0$.
4. $x^3+3x^2+4x-1=0$.
5. $x^3+2x^2+3x-50=0$.
6. $x^3+2x^2-23x-70=0$.
7. $x^4+3x^3-12x^2-20x+53=0$.

8. $x^4 + 8x^3 + 24x^2 + 32x + 16 = 0.$

9. $x^4 - 15x^3 + 77x^2 - 153x + 88 = 0.$

10. $x^4 + 5x^3 - 15x^2 - 77x - 10 = 0.$

Locate between two integers each of the real roots of the following equations:

11. $x^3 - 6x^2 + 7x - 3 = 0.$

12. $x^3 - 8x^2 + 13x - 11 = 0.$

13. $x^3 - 87x + 12 = 0.$

14. $x^3 - 10x - 13 = 0.$

15. $x^3 - 29x + 50 = 0.$

16. $x^3 - 9x^2 + 24x - 17 = 0.$

17. $x^3 - 9x^2 + 23x - 14 = 0.$

18. $x^4 - 8x^3 + 14x^2 + 4x - 8 = 0.$

19. $x^4 + 4x^3 - 4x^2 - 11x + 4 = 0.$

20. $x^4 - 7x^3 + 4x^2 + 20 = 0.$

RATIONAL ROOTS

151. THEOREM VII. *An equation in the form*

$$x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_n = 0,$$

in which b_1, b_2, $\cdots b_n$ are integers, cannot have a rational fraction as a root.

For suppose $\frac{p}{q}$, a fraction in its lowest terms, could be a root of this equation. Substituting $\frac{p}{q}$ for x throughout the equation, we have the identity

$$\frac{p^n}{q^n} + b_1\frac{p^{n-1}}{q^{n-1}} + b_2\frac{p^{n-2}}{q^{n-2}} \cdots + b_n = 0.$$

Multiplying by q^{n-1} and transposing,

$$\frac{p^n}{q} = -b_1p^{n-1} - b_2p^{n-2}q - \cdots - b_nq^{n-1}.$$

Since q cannot be a factor of p^n, we have a fraction equal to an integer, which is impossible. Hence $\frac{p}{q}$ cannot be a root of the given equation and all of its rational roots must be integers.

Let the student write out the proof of the following theorem:

THEOREM VIII. *If* $\frac{p}{q}$, *a fraction in its lowest terms, is a root of the equation* $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0$, q *is a factor of* a_0.

IMAGINARY ROOTS

152. THEOREM IX. *If* $a+bi$ *is a root of* $f(x) = 0$, $a - bi$ *is also a root of that equation.*

If $a+bi$ and $a-bi$ are roots of $f(x)=0$, $x-(a+bi)$ and $x-(a-bi)$ are factors of $f(x)$, and it is, therefore, divisible by their product $(x-a)^2+b^2$. To prove our theorem, it is then sufficient to show that $f(x)$ is divisible by $(x-a)^2+b^2$, when $f(a+bi)=0$.

Since the remainder after dividing $f(x)$ by $(x-a)^2+b^2$ cannot be of higher degree than the first, we may write

$$f(x)=0\,[(x-a)^2+b^2]+cx+d.$$

Since this is an identity, it must be satisfied by $x=a+bi$.

Substituting this value and noting that $f(a+bi)=0$, and that the parenthesis vanishes, since one of its factors reduces to zero, we have

$$ca + cbi + d = 0.$$

Then, by § 29, $cb = 0$, or $c = 0$,

and $ca + d = 0$, or $d = 0$.

Hence $f(x)$ is divisible by $(x-a)^2 + b^2$ and $a-bi$ is a root of the equation $f(x) = 0$.

Since all imaginary roots must appear in pairs, *every equation of odd degree has at least one real root.*

In constructing the graph of an equation, if the curve form an elbow without crossing the X-axis, there will always correspond two imaginary roots. It is not, however, always true that such an elbow can be found for every pair of imaginary roots.

SURD ROOTS

153. Theorem X. *If* $a+\sqrt{b}$ *is a root of the equation* $f(x)=0$, $a-\sqrt{b}$ *is also a root of that equation.*

The proof of this theorem is nearly identical with that of the preceding and should be written out by the student.

RELATIONS BETWEEN THE ROOTS AND COEFFICIENTS

154. In § 52 it was shown that, in the quadratic equation $x^2+px+q=0$, whose roots are α_1 and α_2,

$$\alpha_1+\alpha_2=-p, \text{ and } \alpha_1\alpha_2=q.$$

Again consider the cubic, $x^3+b_1x^2+b_2x+b_3=0$. If its roots are α_1, α_2, and α_3, by § 144,

$$\begin{aligned} x^3+b_1x^2+b_2x+b_3 &= (x-\alpha_1)(x-\alpha_2)(x-\alpha_3) \\ &= x^3-(\alpha_1+\alpha_2+\alpha_3)x^2+(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)x-\alpha_1\alpha_2\alpha_3. \end{aligned}$$

Since this is an identity, the coefficients of like powers of x must be equal, or

$$\alpha_1+\alpha_2+\alpha_3=-b_1,$$

$$\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3=b_2,$$

and

$$\alpha_1\alpha_2\alpha_3=-b_3.$$

It should be noted that in both these cases the sum of the roots is the negative of the coefficient of the second term; that the sum of the roots taken two at a time is the coefficient of the third term; and that, in the cubic, the product of the three roots is the negative of the coefficient of the third term.

These are special cases of the general theorem which we shall now prove.

If $f_n(x)$ be divided by a_0, the coefficient of x^n becomes unity and the general equation of degree n may be written in the form,

$$x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_{n-1}x + b_n = 0.$$

If $\alpha_1, \alpha_2, \cdots \alpha_n$ are the roots of this equation, by § 144

$$x^n + b_1x^{n-1} + \cdots + b_n = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)\cdots(x - \alpha_n).$$

Perform the multiplication indicated in the second member, and it may be seen that the coefficient of x^{n-1} is the negative of the sum of the roots; the coefficient of x^{n-2} is the sum of the roots taken two at a time, etc. Since this is an identity, the coefficients of like powers of x are equal, or

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n &= -b_1, \\ \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 + \cdots \alpha_{n-1}\alpha_n &= b_2, \\ \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_2\alpha_3\alpha_4 + \cdots \alpha_{n-1}\alpha_{n-2}\alpha_n &= -b_3, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \alpha_1\alpha_2\alpha_3\alpha_4\cdots\alpha_n &= (-1)^n b_n. \end{aligned}$$

Although these results are most easily remembered by means of the above equations, we shall state them in the form of a general theorem.

THEOREM XI. *In any equation in which the coefficient of x^n is unity the sum of the roots is equal to the negative of the coefficient of x^{n-1}; the sum of the roots taken two at a time is equal to the coefficient of x^{n-2}; the sum of the roots taken* *three at a time is equal to the negative of the coefficient of* x^{n-3}, *etc.; the product of the roots is equal to the constant term, with the positive or negative sign according as n is even or odd.*

For example, in the equation $5x^3 - 6x + 7 = 0$ we know that the sum of the roots is zero, the product taken two at a time is $-\frac{6}{5}$, and the product of the three roots is $-\frac{7}{5}$.

If there is any known relation existing between the roots of an equation, this theorem often enables us to solve the equation completely.

For example, if in the equation $x^3 - 7x + 6 = 0$ it is known that one root is double another, the roots may be represented by α, 2α, and β.

Then
$$\alpha + 2\alpha + \beta = 3\alpha + \beta = 0,$$
$$\alpha \cdot 2\alpha + \alpha\beta + 2\alpha\beta = 2\alpha^2 + 3\alpha\beta = -7,$$
$$\alpha \cdot 2\alpha \cdot \beta = 2\alpha^2\beta = -6.$$

Solving the first and second equations, $\alpha = \pm 1$, $\beta = \mp 3$. Substituting these values in the third equation, we find that $\alpha = 1$ and $\beta = -3$ satisfy it, but that $\alpha = -1$, $\beta = 3$ do not. Hence the roots are 1, 2, and -3.

EXERCISE CIII

Form the equations which have the following roots, by the aid either of the theorem of § 144 or that of § 154:

1. $1, 2, 3, 4.$
2. $-6, 3, -1.$
3. $3, 3, 2, 2.$
4. $2 - 3i,\ 2 + 3i,\ 2,\ 3.$
5. $7, 4, -11.$
6. $2, 5, \pm\sqrt{-3}.$
7. $-1, \dfrac{1 \pm \sqrt{-3}}{2}.$
8. $\dfrac{1 \pm \sqrt{-3}}{2}, \dfrac{-1 \pm \sqrt{-3}}{2}.$
9. $5, 4, 2, 1.$
10. $10, 8, 5, 1 \pm \sqrt{-3}.$

11. Knowing that one root of the equation $x^4 - 27x^2 + 90x - 36 = 0$ is $3 + \sqrt{-3}$, find the remaining roots.

12. Knowing that one root of the equation $x^4 - 43x^2 - 24x + 108 = 0$ is $4 + \sqrt{7}$, find the remaining roots.

13. Knowing that one root of the equation $x^4 + 5x^3 - 25x^2 - 140x - 26 = 0$ is $-5 + \sqrt{-1}$, find the remaining roots.

14. Solve the equation $x^3 - 3x^2 + 4 = 0$, knowing that two of its roots are equal.

15. Solve the equation $x^3 - 9x^2 + 23x - 15 = 0$, knowing that its roots are in arithmetical progression.

16. Solve the equation $x^4 + 15x^3 + 70x^2 + 120x + 64 = 0$, knowing that its roots are in geometrical progression.

155. To multiply the roots by a constant. In determining fractional and irrational roots, it will be necessary to obtain an equation whose roots are multiples of the roots of a given equation. Let the given equation be

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0,$$

and let it be required to find an equation whose roots are k times the roots of the given equation. This will be obtained by writing the equation formed by substituting $\frac{y}{k}$ for x in the given equation, or

$$a_0\frac{y^n}{k^n} + a_1\frac{y^{n-1}}{k^{n-1}} + a_2\frac{y^{n-2}}{k^{n-2}} + \cdots + a_n = 0.$$

For, if any quantity α satisfies the given equation, then $k\alpha$ must satisfy the second equation, since the result of the substitution will be the same in both. Multiplying this equation by k^n, we have

$$a_0y^n + ka_1y^{n-1} + k^2a_2y^{n-2} + \cdots + k^na_n = 0.$$

From which we have the following

RULE. *To multiply the roots of an equation of degree* n *by* k, *multiply the coefficient of* x^{n-1} *by* k, *of* x^{n-2} *by* k^2, *etc.*

In applying this rule care must be taken to supply all missing terms by writing them with zero as a coefficient.

Ex. 1. Multiply the roots of $x^4 - 3x^3 + x - 5 = 0$ by 3.

Multiplying the second term by 3, the third by 27 (since the term in x^2 is wanting), and the last by 81, we have $x^4 - 9x^3 + 27x - 405 = 0$.

Ex. 2. Transform the equation $8x^3 - 5x + 9 = 0$ into another, whose roots are multiples of the roots of the given equation, and in which the leading coefficient is unity and the remaining coefficients are integers.

Dividing by 8, $\quad x^3 - \frac{5}{8}x + \frac{9}{8} = 0,$

multiplying the roots by k, $\quad x^3 - \frac{5}{8}k^2x + \frac{9}{8}k^3 = 0.$

By inspection we see that the smallest value of k which will make the coefficients integers is 4. Substituting $k = 4$, the equation becomes $x^3 - 10x + 72 = 0$. If the roots of this transformed equation can be found, the roots of the original equation may be determined by dividing these by 4.

156. To change the signs of the roots. In the previous article we have obtained a rule for multiplying the roots of an equation by any constant k. If $k = -1$, the process is equivalent to changing the signs of all the roots. According to this rule the second term must be multiplied by -1, the third by $+1$, the fourth by -1, etc.

Hence the following

RULE. *To form the equation whose roots are the negative of the roots of a given equation, change the sign of every alternate term, the missing terms being supplied with zero coefficients.*

Ex. Change the signs of the roots of the equation $x^4 - 12x^3 + 5x + 4 = 0$.

Changing the signs of $-12x^3$ and $+5x$ (since the term in x^2 is wanting), we have $x^4 + 12x^3 - 5x + 4 = 0$ as the equation whose roots are the negatives of the roots of the given equation.

EXERCISE CIV

1. Multiply the roots of :

(*a*) $x^2 + 6x - 8 = 0$ by 5,

(*b*) $x^4 - 4x^3 + 5x^2 + 11 = 0$ by 4,

(*c*) $x^5 + 3x^4 + 5x^2 - 7 = 0$ by -2.

2. Change the signs of the roots of :

(*a*) $3x^4 + 2x^3 - x^2 + 5x - 10 = 0$,

(*b*) $x^4 - 5x^2 + 2x + 8 = 0$,

(*c*) $3x^5 + x^3 + 4x = 0$.

3. Transform the following equations into others, whose roots are multiples of the roots of the given equations, and in which the leading coefficient is unity and the remaining coefficients are integers :

(*a*) $3x^4 - 2x^3 + 7x^2 + 4 = 0$, (*d*) $10x^4 + 5x^3 + 2x - 8 = 0$,

(*b*) $5x^3 - 6x^2 - 3x + 8 = 0$, (*e*) $7x^4 - 13x^2 - 43 = 0$,

(*c*) $2x^5 - 3x^2 + 10 = 0$, (*f*) $3x^5 - 8x^3 + 3x - 1 = 0$.

4. Find all the rational roots of the following equations by transforming them as in the previous problem, solving the resulting equation by § 146, and dividing the roots obtained by the number used as a multiplier in the first operation:

(*a*) $12x^3 + 8x^2 - 3x - 2 = 0,$

(*b*) $24x^3 + 46x^2 + 9x - 9 = 0,$

(*c*) $18x^3 + 81x^2 + 121x + 60 = 0,$

(*d*) $16x^4 - 48x^3 + 32x^2 + 12x - 9 = 0,$

(*e*) $4x^3 + 12x^2 - 63x + 54 = 0,$

(*f*) $12x^4 - 32x^3 + 13x^2 + 8x - 4 = 0,$

(*g*) $8x^4 - 52x^3 + 66x^2 + 77x - 49 = 0,$

(*h*) $9x^3 + 12x^2 - 11x + 2 = 0.$

157. Variations in sign. A *variation in sign* is said to occur in $f(x)$ when two consecutive terms have opposite signs. Thus, $x^3 - 3x^2 - x + 1$ contains two variations, one in the first and second terms and one in the third and fourth.

THEOREM XII. *If α is a positive root of $f(x) = 0$, the quotient obtained by dividing $f(x)$ by $x - \alpha$ contains at least one less variation than $f(x)$.*

Consider first a numerical example. Divide $x^5 + 3x^4 - 20x^3 - 4x^2 + 32x - 6$ by $x - 3$, performing the operation by synthetic division.

$$\begin{array}{rrrrrr|l} 1 & +3 & -20 & -4 & +32 & -6 & \underline{3} \\ & +3 & +18 & -6 & -30 & +6 & \\ \hline 1 & +6 & -2 & -10 & +2 & & \end{array}$$

Since 3 is positive, the terms of the quotient must remain positive until the sign of the dividend becomes negative. In the example the sign of the quotient changes to negative at the third term. The further terms of the quotient must remain negative until the sign in the dividend becomes positive, when they may or may not change to positive. In the example the fifth term of the quotient is positive. Since the division is exact, the last term in the dividend, -6, must have its sign contrary to the sign of the last term of the quotient, $+2$, and there is one less variation in the quotient than in the dividend.

The general proof of the theorem may be stated in much the same language. As long as the terms of the dividend remain positive, the terms of the quotient must be positive, since each term of the quotient is obtained by the addition of two positive quantities. When there is a change of sign in the dividend, there may be a change of sign in the quotient; since, until such change occurs, the terms of the quotient are obtained by the addition of a positive and a negative quantity. But if this change occurs, since the terms of the quotient are now the sums of two negative quantities, they must remain negative until the sign in the dividend again changes. This process will be repeated for every change in sign of the dividend, and there can be no change in sign in the quotient without a corresponding change in the dividend. But since the division is exact, the sign of the last term of the dividend must be contrary to that of the last term of the quotient, and to the last change of sign in the dividend there can be no corresponding change in the quotient. Hence the theorem.

It is evident from the following example that such a division may decrease the number of variations by more than one.

Divide $x^5 + 3x^4 - 15x^3 - 4x^2 + 2x - 51$ by $x - 3$.

$$\begin{array}{rrrrrr|l} 1 & +3 & -15 & -4 & +2 & -51 & \underline{3} \\ & +3 & +18 & +9 & +15 & +51 & \\ \hline 1 & +6 & +3 & +5 & +17 & & \end{array}$$

Here the number of variations has been decreased by three. The theorem only states that the number of variations must be decreased by *at least* one.

DESCARTES' RULE OF SIGNS

158. Theorem XIII. *An equation $f(x) = 0$ cannot have more positive roots than there are variations in sign in $f(x)$.*

If the equation $f(x) = 0$ has k positive roots, $f(x)$ is divisible by k factors of the form $x - \alpha$. By such division the number of variations in sign in $f(x)$ must be reduced by at least k. Hence the number of variations in sign in $f(x)$ must be at least as large as the number of positive roots.

As we have seen in the last article that each such division may reduce the number of variations by more than one, this theorem does not give the actual number of positive roots, but only a superior limit for that number.

A superior limit for the number of negative roots of $f(x) = 0$ may be obtained by changing the signs of the roots of the given equation by the rule of § 156 and determining the possible number of positive roots of the resulting equation. Hence the following theorem:

Theorem XIV. *An equation $f(x) = 0$ cannot have more negative roots than there are variations in sign in $f(-x)$.*

In case the sum of the possible number of positive and negative roots is less than n, the remaining roots must be imaginary. This can only occur in an equation which lacks some of the powers of x. But this test only gives an inferior limit to the possible number of imaginary roots, since the actual number of positive or negative roots may be less than the limit given by the number of variations.

EXERCISE CV

1. Determine the possible number of positive and negative roots of each of the following equations:

(*a*) $x^4 - 6x^3 + 3x^2 - 6x + 5 = 0$,

(*b*) $x^4 + 6x^2 - 7x + 8 = 0$,

(*c*) $x^5 + x^4 + x^3 - x^2 - x + 1 = 0$,

(*d*) $x^5 - 3x^2 + 6x + 5 = 0$,

(*e*) $4x^3 - x + 5 = 0$,

(*f*) $x^6 + x^4 + x^2 + 1 = 0$.

2. Show that the equation $x^5 - x^4 + x + 1 = 0$ has at least two imaginary roots.

3. Show that the equation $2x^7 - x^4 + 4x - 5 = 0$ has at least four imaginary roots.

4. Show that the equation $x^5 - 1 = 0$ has four imaginary roots.

5. Show that an equation, which has only even powers of x and in which all the coefficients are positive, can have no real roots.

6. Show that every equation of even degree, in which the constant term is negative, must have at least two real roots.

TO DIMINISH THE ROOTS OF AN EQUATION

159. In finding the approximate values of the irrational roots of any equation it will be necessary to find an equation whose roots are equal to the roots of the given equation, each diminished by the same constant. We shall now proceed to establish a rule by which this may be accomplished. Let

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0 \qquad (1)$$

be an equation whose roots are $\alpha_1, \alpha_2, \cdots \alpha_n$. We wish an equation whose roots are $\alpha_1 - k, \alpha_2 - k, \cdots \alpha_n - k$. This will be

$$\begin{aligned} f(z+k) = a_0(z+k)^n + a_1(z+k)^{n-1} + a_2(z+k)^{n-2} \\ + \cdots + a_{n-1}(z+k) + a_n = 0. \qquad (2) \end{aligned}$$

For, if $\alpha_i - k$ is substituted for z in $f(z+k)$, we have $f(\alpha_i)$, or $a_0\alpha_i^n + a_1\alpha_i^{n-1} + a_2\alpha_i^{n-1} + \cdots a_{n-1}\alpha_i + a_n$, which is equal to zero, since α_i is a root of $f(x) = 0$.

Develop each of the powers of $(z+k)$ in equation (2) by the binomial theorem and add. The resulting equation may be reduced to the form

$$A_0z^n + A_1z^{n-1} + A_2z^{n-2} + \cdots A_{n-1}z + A_n = 0. \qquad (3)$$

The values of $A_1, A_2, \cdots A_n$ may be obtained by direct substitution as indicated above; but the following method will be found to be much shorter. Let us reverse the substitution by replacing z in equation (3) by $x - k$. This substitution gives

$$\begin{aligned} A_0(x-k)^n + A_1(x-k)^{n-1} + A_2(x-k)^{n-2} \\ + \cdots + A_{n-1}(x-k) + A_n = 0. \qquad (4) \end{aligned}$$

Since this substitution is simply the reverse of the one made above, equation (4) must be identical with (1). If now we divide equation (4), or its equivalent (1) by $x-k$, the remainder is A_n, and the quotient is

$$A_0(x-k)^{n-1}+A_1(x-k)^{n-2}+A_2(x-k)^{n-3}+\cdots+A_{n-1}.$$

Again, dividing this quotient by $x-k$, the remainder is A_{n-1}, and this process may be continued until we have determined all the coefficients of equation (3). Hence we have the following rule for diminishing the roots of an equation $f(x)=0$ by a constant k:

RULE. *Divide* $f(x)$ *by* $x-k$, *the quotient by* $x-k$, *and so on; the last quotient and the remainders in reverse order will be the coefficients of* x^n, x^{n-1}, *etc.*

Ex. 1. Diminish the roots of $3x^4-12x^3+6x^2+10x-12=0$ by 3.

The process of dividing $f(x)$ by $x-3$, the quotient by $x-3$, etc., may be arranged as follows:

$$\begin{array}{rrrrrll}
3 & -12 & +6 & +10 & -12 & \underline{|\,3} & \\
 & +9 & -9 & -9 & +3 & & \\
\cline{1-5}
3 & -3 & -3 & +1 & \mathbf{-9} & & \therefore A_4=-9. \\
 & +9 & +18 & +45 & & & \\
\cline{1-5}
3 & +6 & +15 & \mathbf{+46} & & & \therefore A_3=46. \\
 & +9 & +45 & & & & \\
\cline{1-4}
3 & +15 & \mathbf{+60} & & & & \therefore A_2=60. \\
 & 9 & & & & & \\
\cline{1-3}
\mathbf{3} & \mathbf{+24} & & & & & \therefore A_1=24,\ A_0=3.
\end{array}$$

Hence the equation each of whose roots is 3 less than the roots of the given equation is

$$3x^4+24x^3+60x^2+46x-9=0.$$

Ex. 2. Increase the roots of $x^3-3x^2+4=0$ by 2.

Since increasing the roots by 2 is equivalent to decreasing them by -2, we must divide synthetically by -2.

$$\begin{array}{rrrrl}
1 & -3 & 0 & +4 & \underline{|-2} \\
 & -2 & +10 & -20 & \\
\hline
1 & -5 & +10 & -16 & \quad \therefore A_3=-16. \\
 & -2 & +14 & & \\
\cline{1-4}
1 & -7 & +24 & & \quad \therefore A_2=24. \\
 & -2 & & & \\
\cline{1-3}
1 & -9 & & & \quad \therefore A_1=-9,\ A_0=1.
\end{array}$$

Hence the required equation is

$$x^3-9x^2+24x-16=0.$$

EXERCISE CVI

1. Diminish the roots of $x^3+8x^2-6x-8=0$ by 1.
2. Diminish the roots of $3x^4+x^3+6x+5=0$ by 3.
3. Diminish the roots of $x^4-3x^2+6=0$ by 5.
4. Increase the roots of $x^3+8x^2-x+6=0$ by 4.
5. Increase the roots of $x^5+5x^4+3x^2-x=0$ by 8.
6. Diminish the roots of $3x^4+5x^3-x+6=0$ by -6.
7. Diminish the roots of $x^5+3x^4+4x^2-5=0$ by 10.
8. Diminish the roots of $x^5-3x^2+8=0$ by 5.
9. Determine all the roots, real and imaginary, of

    (*a*) $x^3-8=0$,
    (*b*) $x^3+1=0$,
    (*c*) $x^4+1=0$,
    (*d*) $x^4+16=0$,
    (*e*) $x^4-16=0$,
    (*f*) $x^6-1=0$,
    (*g*) $x^6-64=0$,
    (*h*) $x^6+1=0$,
    (*i*) $x^8-1=0$,
    (*j*) $x^8-256=0$.

COMPUTATION OF IRRATIONAL ROOTS BY HORNER'S METHOD

160. We have now proved all the theorems which are needed for the approximate calculation of the irrational roots of any equation. The simplest method of obtaining these roots is illustrated by the following example.

Find the smallest positive root of $x^3-3x^2-4x+10=0$.

By Descartes's rule of signs we see that there may be two positive roots and one negative root. By the method of § 147 we obtain the following table of values and graph:

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
$x=$	-3	-2	-1	0	1	2	3	4	5
$f(x)=$	-32	-2	10	10	4	-2	-2	10	40

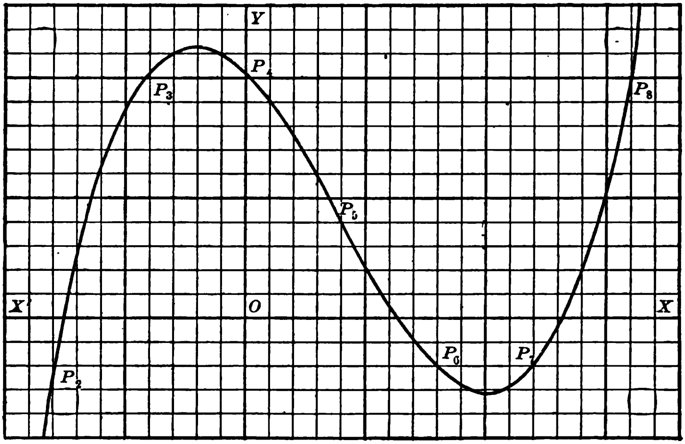


FIG. 17.

In constructing the graph four horizontal units represent a unit in the value of x, while each vertical unit represents a single unit in the value of $f(x)$.

Since the graph crosses the X-axis between -2 and -1, 1 and 2, and between 3 and 4, there are roots of the equation between each of these values of x. We are to obtain the approximate value of that root which lies between 1 and 2. Suppose this root is $1.abcd\cdots$, where a, b, c, d are the digits in the decimal part of the root. The first step in the process is to obtain an equation whose roots are equal to the roots of the given equation each diminished by 1. By the method of § 159 this may be accomplished as follows:

$$\begin{array}{rrrr|l}
1 & -3 & -4 & +10 & \underline{1} \\
 & 1 & -2 & -6 & \\ \hline
1 & -2 & -6 & +4 & \\
 & 1 & -1 & & \\ \hline
1 & -1 & -7 & & \\
 & 1 & & & \\ \hline
1 & 0 & & &
\end{array}$$

The resulting equation is

$$x^3-7x+4=0. \qquad (1)$$

Since one root of the given equation was $1.abcd\cdots$, the corresponding root of (1) is the decimal $.abcd\cdots$ The second step in the process is to multiply the roots of (1) by 10 to avoid the use of decimals.

The result is $$x^3-700x+4000=0, \qquad (2)$$

and one root of (2) is $a.bcd\cdots$. This root must lie between 0 and 10, and by the method of § 150 we may locate it between two digits and thus determine the value of a. Substituting 6,

$$\begin{array}{rrrr|l}
1 & 0 & -700 & +4000 & \underline{6} \\
 & 6 & 36 & -3984 & \\ \hline
 & 6 & -664 & +\ \ 16 &
\end{array}$$

we find $f(6)=+16$. Substituting 7,

$$\begin{array}{rrrr|l}
1 & 0 & -700 & +4000 & \underline{7} \\
 & 7 & 49 & -4557 & \\ \hline
 & 7 & -651 & -\ \ 557 &
\end{array}$$

we find $f(7)=-557$. There is, therefore, a root between

6 and 7, and we have obtained the value 6 for *a*, the second digit in the root.

Proceeding as before, reduce the roots of equation (2) by 6.

$$\begin{array}{rrrrl}
1 & 0 & -700 & +4000 & \underline{|6} \\
 & 6 & 36 & -3984 & \\
\hline
1 & 6 & -664 & +\ \ \mathbf{16} & \\
 & 6 & 72 & & \\
\cline{1-3}
1 & 12 & \mathbf{-592} & & \\
 & 6 & & & \\
\cline{1-2}
\mathbf{1} & \mathbf{18} & & &
\end{array}$$

The resulting equation is

$$x^3+18x^2-592x+16=0, \qquad (3)$$

one of whose roots is $.bcd \cdots$. Again, multiplying the roots of this equation by 10, we have

$$x^3+180x^2-59200x+16000=0, \qquad (4)$$

one of whose roots is $b.cd \cdots$. Since $f(0)=16000$, and $f(1)=-43019$, there is a root of equation (4) between 0 and 1. Hence $b=0$, and the approximate value of the root of the given equation which we have now found is 1.60.

Since $b=0$, we may proceed at once to multiply the roots of (4) by 10. The resulting equation is

$$x^3+1800x^2-5920000x+16000000=0, \qquad (5)$$

one of whose roots is $c.d \cdots$.

We have now reached a point in the solution where the last two terms in the equation are so large in comparison with the first two that at least one and probably two more figures of the root may be obtained by disregarding the first two terms and dividing the constant term by the coefficient of x. This gives 2.7, and the root as now found is 1.6027. Just at what point in the work it is safe to discard the highest powers of x and obtain one or more figures of the root by dividing the constant term by the coefficient of x, depends upon the nature of the problem. Usually after the first figure in the decimal part of the root has been found one figure may be found by such division and in this way the labor of substitution may be avoided. Even as early in the work as the determination of the first digit of the decimal it may be of assistance, as may be seen in our example. In equation (2) if x^3 is discarded, the root is seen to be in the neighborhood of 6. But in determining whether it is less or greater than 6, we must find $f(6)$ and $f(7)$ as in the solution. Again, in equation (4), by using the last two terms only we see that the root must be less than 1.

161. To illustrate the method further we shall find the negative root of the same equation. Forming the equation whose roots are the negatives of the roots of the given equation, we have

$$x^3 + 3x^2 - 4x - 10 = 0. \tag{1}$$

Since there was a root of the given equation between -1 and -2, there will be a root of (1) between 1 and 2. Let this root be represented by $1.abcd \cdots$. Decrease the roots of (1) by 1.

$$\begin{array}{rrrr}
1 & +3 & -4 & -10 \\
 & 1 & 4 & 0 \\
\hline
1 & 4 & 0 & \mathbf{-10} \\
 & 1 & 5 & \\
\cline{1-3}
1 & 5 & \mathbf{5} & \\
 & 1 & & \\
\cline{1-2}
\mathbf{1} & \mathbf{6} & &
\end{array}$$

The resulting equation is

$$x^3+6x^2+5x-10=0, \qquad (2)$$

one of whose roots is $.abcd\cdots$. Multiplying the roots of (2) by 10, we have

$$x^3+60x^2+500x-10000=0, \qquad (3)$$

one of whose roots is $a.bcd\cdots$. Here the first two terms are so large in comparison with the others that, if they are discarded, we shall obtain no indication of the value of a except that it is large. By synthetic division we find $f(8)=-1648$ and $f(9)=89$. Hence $a=8$. Decreasing the roots of (3) by 8,

$$\begin{array}{rrrr}
1 & 60 & 500 & -10000 \\
 & 8 & 544 & 8352 \\
\hline
1 & 68 & 1044 & \mathbf{-1648} \\
 & 8 & 608 & \\
\cline{1-3}
1 & 76 & \mathbf{1652} & \\
 & 8 & & \\
\cline{1-2}
\mathbf{1} & \mathbf{84} & &
\end{array}$$

and multiplying the roots by 10, the resulting equation is

$$x^3+840x^2+165200x-1648000=0, \qquad (4)$$

one of whose roots is $b.cd\cdots$.

We may now obtain a pretty clear indication from the last two terms that the value of c is 9, and this is confirmed when we find, on decreasing the roots by 9, that the constant term remains negative. Decreasing the roots by 9, and multiplying the roots of the resulting equation by 10, we find

$$x^3+8670x^2+18056300x-102431000.$$

Two more figures of the root may now be found by dividing the last term by the coefficient of x, or 1024 by 181, which gives 5.6. Hence the negative root of the given equation is $-1.8956+$.

162. As a further illustration we shall plot the graph of the equation, $x^4+2x^3-12x^2-x+2=0$, and find its largest root.

The table of values is as follows :

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
$x=$	-5	-4	-3	-2	-1	0	1	2	3
$f(x)=$	82	-58	-76	-44	-10	2	-8	-16	26

In constructing the graph two horizontal units represent a unit in the value of x, while each vertical unit represents eight units in the value of $f(x)$.

There are seen to be roots of the equation between -4 and -5, -1 and 0, 0 and 1, 2 and 3. We wish to obtain that root which lies between 2 and 3. We must, therefore, diminish the roots of the equation by 2.

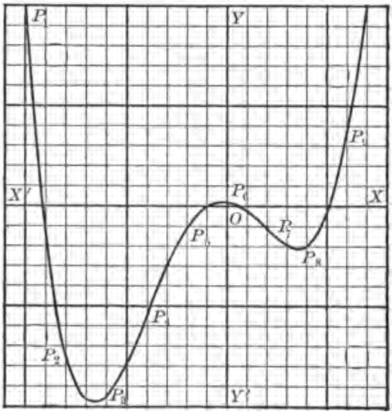


FIG. 18.

$$
\begin{array}{rrrrr|l}
1 & +2 & -12 & -1 & +2 & \underline{2} \\
 & 2 & -8 & -8 & -18 & \\
\hline
1 & 4 & -4 & -9 & \mathbf{-16} & \\
 & 2 & 12 & 16 & & \\
\hline
1 & 6 & 8 & \mathbf{7} & & \\
 & 2 & 16 & & & \\
\hline
1 & 8 & \mathbf{24} & & & \\
 & 2 & & & & \\
\hline
\mathbf{1} & \mathbf{10} & & & &
\end{array}
$$

The resulting equation is

$$x^4 + 10x^3 + 24x^2 + 7x - 16 = 0.$$

Multiplying its roots by 10,

$$x^4 + 100x^3 + 2400x^2 + 7000x - 160000 = 0.$$

Since $f(6) = -8704$ and $f(7) = 43301$, there is a root of this equation between 6 and 7. Decreasing the roots of the equation by 6,

$$\begin{array}{rrrrr|l}
1 & 100 & 2400 & 7000 & -160000 & \underline{6} \\
 & 6 & 636 & 18216 & 151296 & \\
\hline
1 & 106 & 3036 & 25216 & \mathbf{-8704} & \\
 & 6 & 672 & 22248 & & \\
\cline{1-4}
1 & 112 & 3708 & \mathbf{47464} & & \\
 & 6 & 708 & & & \\
\cline{1-3}
1 & 118 & \mathbf{4416} & & & \\
 & 6 & & & & \\
\cline{1-2}
\mathbf{1} & \mathbf{124} & & & &
\end{array}$$

The resulting equation is

$$x^4 + 124x^3 + 4416x^2 + 47464x - 8704 = 0.$$

Multiplying its roots by 10,

$$x^4 + 1240x^3 + 441600x^2 + 47464000x - 87040000 = 0.$$

Dividing the last term by the coefficient of x, we see that there is a root between 1 and 2. Decreasing the roots by 1,

$$\begin{array}{rrrrr|l}
1 & 1240 & 441600 & 47464000 & -87040000 & \underline{1} \\
 & 1 & 1241 & 442841 & 47906841 & \\
\hline
1 & 1241 & 442841 & 47906841 & \mathbf{-39133159} & \\
 & 1 & 1242 & 444083 & & \\
\cline{1-4}
1 & 1242 & 444083 & \mathbf{48350924} & & \\
 & 1 & 1243 & & & \\
\cline{1-3}
1 & 1243 & \mathbf{445326} & & & \\
 & 1 & & & & \\
\cline{1-2}
\mathbf{1} & \mathbf{1244} & & & &
\end{array}$$

Multiplying the roots of the resulting equation by 10 and dividing the last term by the coefficient of x, we find 8 as the next figure in the root. Here the coefficient of x^2 is so large that we can be sure of only one figure by this division and, if we wish to know the fourth decimal place accurately, the complete operation must be performed again. The root is 2.618^+.

EXERCISE CVII

Compute each of the real roots of the following equations correct to the third decimal figure:

1. $x^3 - 9x + 5 = 0.$
2. $x^3 - 2x - 5 = 0.$
3. $x^3 - 29x + 54 = 0.$
4. $x^3 + x^2 + x - 100 = 0.$
5. $x^3 + x^2 - 2x - 1 = 0.$
6. $4x^3 - 13x^2 - 31x - 275 = 0.$
7. $x^4 + 2x^3 - 12x^2 - x + 2 = 0.$
8. $x^4 + 5x^3 - 15x^2 - 77x - 10 = 0.$
9. $x^4 - 46x^2 - 7x + 2 = 0.$
10. $x^4 - 7x^3 + 4x^2 + 20 = 0.$
11. $x^4 - 4x^3 - 4x^2 - 11x + 4 = 0.$
12. $x^4 - 12x^2 + 12x - 3 = 0.$
13. $x^5 - 37x^3 + x^2 - 30x + 5 = 0.$
14. $x^5 - 6x^4 + x^3 + 13x^2 + 138x - 315 = 0.$
15. $x^6 - 38x^4 - 40x^3 + 217x^2 + 160x - 215 = 0.$

CHAPTER XIX

GRAPHICAL REPRESENTATION OF COMPLEX NUMBERS

163. In Chapter III it was shown that all real numbers, rational and irrational, may be represented by points on a line; in this chapter will be described the method of representing in a similar way all complex numbers by points in a plane.

Construct a pair of perpendicular axes just as in the ordinary graphical representation of equations. To represent the complex number $x+yi$, locate the point (x, y) in the plane in the usual way by measuring x units from the Y-axis, to the right if x is positive, to the left if x is negative; and y units from the X-axis, upward if y is positive, downward if y is negative. The point (x, y) may then be used to represent the complex number $x+yi$. It is easily seen that in this way every complex number may be represented by a single point in the plane, which is not the same for any two unequal complex numbers; and conversely, that to every point in the plane will correspond one and only one complex number.

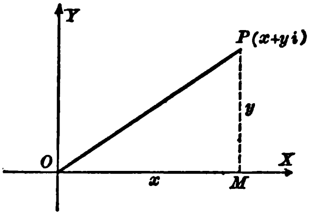


FIG. 19.

Since $y=0$ for all real numbers, such numbers will always be represented by points on the X-axis. Hence it is called the **axis of reals**. Also, $x=0$ for all pure imaginaries, and such numbers are therefore represented by points on the Y-axis, which is called the **axis of imaginaries.**

If both the direction and magnitude of the line OP are considered, there will be a different line for every complex number, and this line also may be said to represent the number. Hence, *either the point P or the line OP may be used to picture the complex number.* The length of the line OP or the distance of P from the origin is called the **absolute** value of the complex number $x+yi$; while the angle XOP is called the **amplitude** of $x+yi$.

164. Addition. In Chapter VI the sum of the two complex numbers $a+bi$ and $c+di$ was defined as $a+c+(b+d)i$. The graphical representation of this addition is very simple.

Let OP_1 represent $a+bi$, and OP_2, $c+di$. Through P_1 draw P_1P_3, a line parallel and equal to OP_2. Then the point P_3 or the line OP_3 represents the sum of $a+bi$ and $c+di$.

For $OM=a$, $MP_1=b$,

$OL=P_1K=MN=c$,

$LP_2=KP_3=d$.

Hence,

$ON=OM+MN=a+c$,

$NP_3=MP_1+KP_3=b+d$,

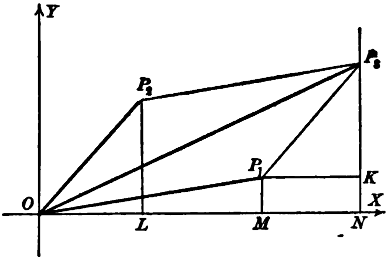


FIG. 20.

and the point P_3 has the coördinates $(a+c, b+d)$, and represents the complex number $a+c+(b+d)i$.

Hence, *the sum of two numbers which are represented by the points P_1 and P_2 is represented by the fourth vertex of the parallelogram formed by drawing lines through P_1 and P_2, parallel to OP_2 and OP_1.*

If we think of the complex numbers as represented by the lines OP_1 and OP_2, their sum is represented by the line OP_3 from the origin to the extremity of a line drawn through P_1, having the same direction as OP_2 and equal to it in length.

To find the point which represents the difference between the numbers represented by P_3 and P_2, draw through P_3 a line P_3P_1, equal in length to OP_2 and opposite to it in direction, and its extremity P_1, or the line OP_1, represents the difference of the two numbers.

EXERCISE CVIII

Plot the points representing each of the following complex numbers and the points which represent their sum or difference:

1. $5i+(2-3i)$.
2. $3+(5+2i)$.
3. $(6-2i)+(3+i)$.
4. $(3-4i)+(6+5i)$.
5. $(2-3i)-(1-i)$.
6. $(7-8i)-(3-4i)$.
7. $(3-i)-(6+5i)$.
8. $i+(8-i)$.
9. $i-(10+i)$.
10. $(6+3i)+(3-5i)$.

165. Polar representation. For the further discussion of this subject some knowledge of trigonometry is necessary.

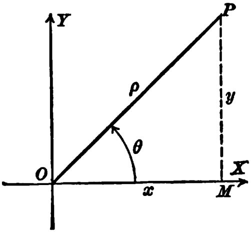


FIG. 21.

In § 163 the length of OP has been defined as the absolute value, and the angle XOP as the amplitude of the number $x+yi$. Let ρ represent the absolute value, and θ the amplitude.

Since

$$\sin\theta = \frac{MP}{OP} \text{ and } \cos\theta = \frac{OM}{OP},\quad x = \rho\cos\theta,\quad y = \rho\sin\theta,$$

and

$$x + yi = \rho(\cos\theta + i\sin\theta).$$

This method of writing the complex number is convenient for many purposes. ρ is always a positive number, but θ may have an infinite set of values differing by 2π; since values of θ differing by multiples of 2π determine the same position for OP. That value of the amplitude which lies between $-\pi$ and $+\pi$ is called the **chief amplitude**. If it is understood that θ shall always be used to represent the chief amplitude, then $x+yi$ may be written

$$\rho[\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)],\ (n = 1, 2, 3 \cdots).$$

166. Multiplication. Multiplying the complex numbers

$$\rho_1(\cos\theta_1 + i\sin\theta_1)$$

and

$$\rho_2(\cos\theta_2 + i\sin\theta_2),$$

we have

$$\rho_1\rho_2[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)].$$

But

$$\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 = \cos(\theta_1 + \theta_2),$$

and

$$\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1 = \sin(\theta_1 + \theta_2).$$

Hence, the product is

$$\rho_1\rho_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)],$$

in which the absolute value is $\rho_1\rho_2$ and the amplitude is $\theta_1 + \theta_2$. That is, the absolute value of the product is the product of the absolute values, and the amplitude of the product is the sum of the amplitudes of the factors.

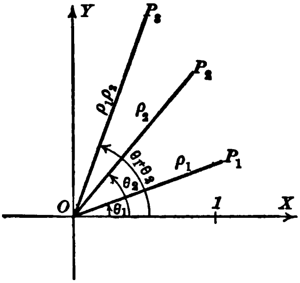


FIG. 22.

If in Fig. 22 P_3 represents the product of the numbers represented by P_1 and P_2, $OP_3 = OP_1 \times OP_2$, and $\angle XOP_3 = \angle XOP_1 + \angle XOP_2$.

Since division is the inverse of multiplication, the absolute value of the quotient of two numbers is the quotient of their absolute values, and the amplitude of the quotient is the difference of their amplitudes.

EXERCISE CIX

1. Find the absolute value and amplitude of

(a) $1+i$, (b) $\sqrt{3}+1$, (c) $1+i\sqrt{2}$,
(d) $1+i\sqrt{3}$, (e) $\sqrt{2}+i$.

2. Plot each of the following complex numbers and determine by plotting the position of the points which represents their products. Compare result with the points obtained by multiplying algebraically and plotting the result.

(a) $(3+2i)\times(5-i)$,
(b) $(1-i)\times(1+i)$,
(c) $i\times(3+2i)$,
(d) $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\times\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$,
(e) $(6-5i)\times(6+5i)$,
(f) $(\frac{7}{3}-2i)\times(\frac{2}{3}-7i)$.

167. Powers and roots. By the rule obtained for multiplication,

$$[\rho(\cos\theta+i\sin\theta)]^2=\rho^2(\cos 2\theta+i\sin 2\theta),$$

and
$$[\rho(\cos\theta+i\sin\theta)]^3=\rho^3(\cos 3\theta+i\sin 3\theta).$$

Similarly, $[\rho(\cos\theta+i\sin\theta)]^n=\rho^n(\cos n\theta+i\sin n\theta)$.

Hence, *the absolute value of the nth power of a complex number is the nth power of its absolute value; and the amplitude of the nth power of a number is n times its amplitude.*

Since extracting a root is the inverse of raising to a power,

$$\sqrt[n]{\rho(\cos\theta + i\sin\theta)} = \sqrt[n]{\rho}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right),$$

for $$\left[\sqrt[n]{\rho}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)\right]^n = \rho(\cos\theta + i\sin\theta).$$

Since the absolute value of a complex number is real and positive, *the absolute value of the nth root of a number is the principal value of the nth root of its absolute value.*

But we have seen (§ 165) that the amplitude of any complex number has an infinite set of values, $\theta + 2k\pi$, $(k = 0, 1, 2, 3 \cdots)$, all of which have the same terminal line. Hence the amplitude of the nth root is $\frac{\theta + 2k\pi}{n} = \frac{\theta}{n} + \frac{k}{n}2\pi$, $(k = 0, 1, 2, \cdots)$, which are seen to have different terminal lines for $k = 0, 1, 2, \cdots n-1$. For values of k larger than $n-1$ the same terminal lines will be obtained, since $\frac{\theta}{n} + \frac{n+k}{n}2\pi$ is congruent with $\frac{\theta}{n} + \frac{k}{n}2\pi$. Hence we find just n different values of the amplitude of the nth root of any number. There are, therefore, n different complex numbers whose nth power is the given number. Since the absolute values of all these numbers are the same, the points which represent them are situated on a circle of radius $\sqrt[n]{\rho}$, and divide this circle into n equal arcs.

EXAMPLES

1. Obtain $\sqrt[3]{i}$ and plot the corresponding points.

Since $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$,

$$\sqrt[3]{i} = \cos\frac{90^\circ + 2k\pi}{3} + i\sin\frac{90 + 2k\pi}{3}, \quad [k = 0, 1, 2]$$

$$= \cos 30^\circ \ + i\sin 30^\circ \ = \tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}i,$$

or $$= \cos 150^\circ + i\sin 150^\circ = -\tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}i,$$

or $$= \cos 270^\circ + i\sin 270^\circ = -i.$$

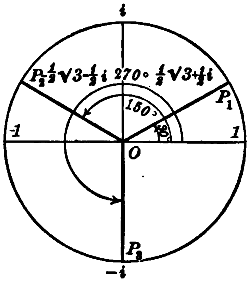

i
P_2 $\frac{1}{2}\sqrt{3}-\frac{1}{2}i$
270°
$\frac{1}{2}\sqrt{3}+\frac{1}{2}i$
P_1
150°
30°
-1
1
O
P_3
$-i$

FIG. 23.

We see that $\sqrt[3]{i}$ has three values, each having the absolute value unity, but differing in amplitude by 120°. They are represented by the three points P_1, P_2, and P_3 in Fig. 23.

2. Obtain $\sqrt{1 + \sqrt{3}\,i}$ and plot the corresponding points.

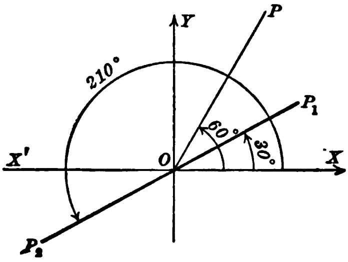


FIG. 24.

Since

$$1+\sqrt{3}\,i=2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\,i\right)$$

$$=2(\cos 60^\circ+i\sin 60^\circ),$$

$$\sqrt{1+\sqrt{3}\,i}=\sqrt{2}\left(\cos\frac{60^\circ+2\,k\pi}{2}\right.$$

$$\left.+\,i\sin\frac{60^\circ+2\,k\pi}{2}\right),\ [k=0,\ 1]$$

$$=\sqrt{2}(\cos 30^\circ+i\sin 30^\circ)$$

$$=\sqrt{2}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\,i\right)=\frac{1}{2}\sqrt{6}+\frac{1}{2}\sqrt{2}\,i,$$

or

$$=\sqrt{2}(\cos 210^\circ+i\sin 210^\circ)$$

$$=-\tfrac{1}{2}\sqrt{6}-\tfrac{1}{2}\sqrt{2}\,i.$$

EXERCISE CX

1. Calculate the absolute value and amplitude of the following fractions; plot the points which represent the numerator, the denominator, and the fraction:

(*a*) $\dfrac{2+i}{3-i}$, (*b*) $\dfrac{5-3i}{3+i}$, (*c*) $\dfrac{1-i}{1+i}$, (*d*) $\dfrac{i}{2+i}$.

2. Obtain all of the roots and plot the corresponding points in

(*a*) $\sqrt{2+i}$, (*b*) $\sqrt[3]{1+i}$, (*c*) $\sqrt[4]{1-i}$, (*d*) $\sqrt[4]{i}$,
(*e*) $\sqrt[5]{2+2\sqrt{3}\,i}$.

3. Find all the roots of the following equations:

(*a*) $x^3-1=0$, (*b*) $x^4+1=0$, (*c*) $x^5+1=0$,
(*d*) $x^5-1=0$, (*e*) $x^7+1=0$, (*f*) $x^7-1=0$.

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